Classical Mechanics - PHYS 310 - Fall 2013 Lec 11: Supplementary Notes
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EXAMPLE: ELECTRICAL ANALOG TO THE MECHANICAL VIBRATIONS

In the class, we discussed, how different electrical components such as inductors, capacitors and resistors as well as external voltage sources are analog to the components of the mechanical vibrations such as: restoring forces, masses, and resistive forces and driving forces.

Using this similarity, we have evaluated the oscillations in LCR circuits. Let's do one more example.

Example A mass m_1 driven by a sinusoidal force whose frequency is ω. The mass $m1$ is attached to a rigid support by a spring of force constant k and slides on a second mass m_2. The frictional force between m_1 and m_2 is represented by the damping parameter b_1. and the frictional force between m_2 and the support is represented by b_2. Construct the electrical analog of this system.

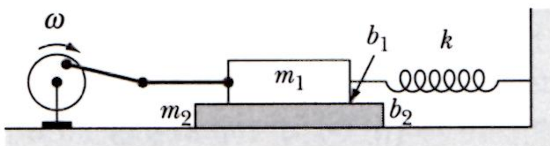


FIGURE 3-B Problem 3-26.

We first need to write down the equation of motion. Let's take the coordinate of the mass m_1 as x_1, and for mass m_2 as x_2.

The forces experience by the mass m_1:

- restoring force from the spring $-kx_1$
- The damping force from the m_1 $m2$ interface: $-b_1(\dot{x}_1 - \dot{x}_2)$
- and the driving force $FCos\omega t$

The forces experience by the mass m_2:

- the damping from the $m1$ and m_2 interface
- the damping force from the horizontal surface

So from our previous knowledge, we can write the equations of motion as:

$$m_1\ddot{x}_1 = -kx_1 - b_1(\dot{x}_1 - \dot{x}_2) + FCos\omega t \tag{1}$$

$$m_2\ddot{x}_2 = -b_2\dot{x}_2 - b_1(\dot{x}_2 - \dot{x}_1) \tag{2}$$

Let's put all the electrical analog as,

$$\begin{aligned} m_1 &\to L_1; \\ m_2 &\to L_2; \\ k &\to \frac{1}{C} \\ b_1 &\to R_1 \\ b_2 &\to R_2 \\ x_1 &\to q_1 \\ x_2 &\to q_2 \end{aligned}$$

Let's simply convert the above two equations of motion with these electrical analogs.

$$L_1\ddot{q}_1 = -\frac{1}{C}q_1 - R_1(\dot{q}_1 - \dot{q}_2) + \epsilon_0 Cos\omega t \tag{3}$$

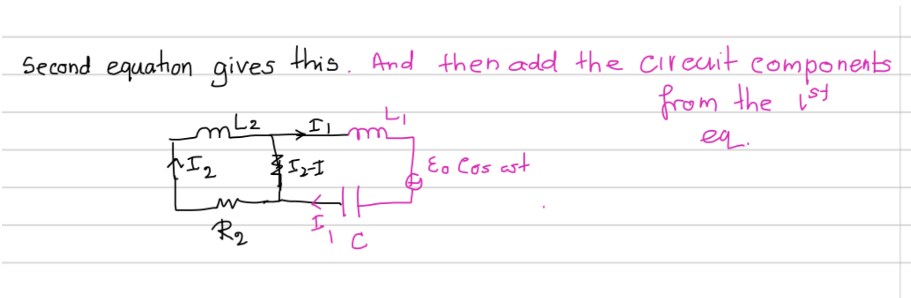
$$L_2\ddot{q}_2 = -R_2\dot{x}_2 - R_1(\dot{q}_2 - \dot{q}_1) \tag{4}$$

Let's re-write this equations:

$$L_1\frac{dI_1}{dt} + \frac{1}{C}q_1 + R_1(I_1 - I_2) = \epsilon_0 Cos\omega t \tag{5}$$

$$L_2\frac{dI_2}{dt} + R_2 I_2 + R_1(I_2 - I_1) = 0 \tag{6}$$

We can out these circuit elements together to form the following circuit.



PRINCIPLE OF SUPERPOSITION

In the previous lecture, we have discussed the harmonic oscillator in the presence of a sinusoidal driving force. The equation we have solved is :

$$\left(\frac{d^2}{dx^2} + 2\beta\frac{d}{dx} + \omega_0^2\right)x(t) = ACos\omega t \tag{7}$$

This is a linear differential equation. In fact,w e can write it as,

$$\mathbf{L}x(t) = F(t) \tag{8}$$

where $\mathbf{L}$ is a linear operator. What that means is If we have two equations like:

$$\mathbf{L}x_1(t) = F_1(t) \tag{9}$$

and

$$\mathbf{L}x_2(t) = F_2(t) \tag{10}$$

We can write this as:

$$\mathbf{L}\left[x_1(t) + x_2(t)\right] = F_1(t) + F_2(t) \tag{11}$$

In fact, we can generalize this equation as:

$$\mathbf{L}\left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right] = \alpha_1 F_1(t) + \alpha_2 F_2(t) \tag{12}$$

We can write the force function as a linear superposition of a series of functions as: $F_{ext} = \sum_n \alpha_n F_n(t)$, where each of these F_n components satisfy the differential equations as:

$$\mathbf{L}\alpha_n x_n(t) = \alpha_n F_n(t) \tag{13}$$

That is, when the force is a linear superposition of a force components, then the solution to the total differential equation can be written as a linear superposition of $x_n(t)$.

$$x(t) = \sum_n \alpha_n x_n(t) \tag{14}$$

That means, when we can write the total force on the SHO problem,we can solve for $x(t)$ for each force component and the combine them to find the $x(t)$ as shown above.

If we have a periodic force function: such that $F(t) = F(t+T)$, we can write the force function in terms of a Fourier series. For instance we can write the force as:

$$F(t) = \sum_n \alpha_n Cos(\omega_n t - \phi_n) \tag{15}$$

We can solve for the steady state for each individual force component and then write the complete steady state solution as:

$$x(t) = \sum_n \frac{\alpha_n/m}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}} Cos(\omega_n t - \phi_n - \delta_n) \tag{16}$$

where δ_n is the phase difference between n^{th} force component and its response:

$$\delta_n = Tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega_n^2}\right) \tag{17}$$

Now, given a complicated force function, if you can divide it to components, by the method of linear superposition, we can solve for the resultant motion.

NON-SINUSODIAL PERIODIC FUNCTIONS

We consider the case of a force which varies as any arbitrary periodic function. According to Fourier theorem, any periodic function can be represented in terms of a series of sinusoidal functions.

Which means:

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n Cos\, n\omega t + b_n Sin\, n\omega t) \tag{18}$$

where,

$$\begin{aligned} a_n &= \frac{2}{\tau}\int_0^{\infty} F(t')Cos\, n\omega t' dt' \\ b_n &= \frac{2}{\tau}\int_0^{\infty} F(t')Sin\, n\omega t' dt' \end{aligned} \tag{19}$$

Once we know the coefficients a_n and b_n, we know the Fourier expansion of the force, then we can use the linear superposition to solve the problem,

Sawtooth Driving Force on the Damped Harmonic Oscillator

A sawtooth driving force function is applied on the SHO with a damping force, Explain how you would solve the equation of motion for this problem.

$F(t)$ can be explained as:

$$F(t) = A\frac{t}{\tau} \quad -\tau/2 < t < \tau/2 \tag{20}$$

Saw tooth function is a odd function. Now that Cosine function is an even function, all the a_n coefficients go to zero. Sin function is an odd functions, so that b_n coefficients are non-zero.

Let's work on and find out the b_n coefficients.

$$\begin{aligned} b_n &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} F(t')Sin\ n\omega t' dt' \\ &= \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} A\frac{t'}{\tau}Sin\ n\omega t' dt' \\ &= \frac{2A}{\tau^2}\int_{-\tau/2}^{\tau/2} t' Sin\ n\omega t' dt' \end{aligned} \tag{21}$$

In order to solve the last equation, we can use the integration by parts $\int udv = uv - \int vdu$
Let's re-write the eq.(21) as:

$$b_n = \frac{2A}{\tau^2}(\frac{-1}{n\omega})\int_{-\tau/2}^{\tau/2} t' dCos\, n\omega t' \tag{22}$$

Using the integration by parts:

$$b_n = \frac{2A}{\tau^2}(\frac{-1}{n\omega})\left[t'Cosn\omega t'|_{-\tau/2}^{\tau/2} - \int Cos\ n\omega t' dt'\right] \tag{23}$$

$$b_n = \frac{2A}{\tau^2}(\frac{-1}{n\omega})\left[t'Cosn\omega t'|_{-\tau/2}^{\tau/2} - \frac{Sin\ n\omega t'}{n\omega}|_{-\tau/2}^{\tau/2}\right] \tag{24}$$

Now we have both τ and ω in these equations. Let's substitute $\tau = 2\pi/\omega$

$$b_n = \frac{2A\omega^2}{(2\pi)^2}(\frac{-1}{n\omega})\left[t'Cosn\omega t'|_{-\pi/\omega}^{\pi/\omega} - \frac{Sin\ n\omega t'}{n\omega}|_{-\pi/\omega}^{\pi/\omega}\right] \tag{25}$$

The second term in the parenthesis goes to zee because $Sin\ n\pi = 0$, which gives:

$$b_n = \frac{2A\omega^2}{(2\pi)^2}(\frac{-1}{n\omega})\left[t'Cosn\omega t'|_{-\pi/\omega}^{\pi/\omega}\right] \tag{26}$$

$$b_n = \frac{2A\omega^2}{(2\pi)^2}(\frac{-1}{n\omega})\left[\frac{2\pi}{\omega}Cos\ n\pi\right] \tag{27}$$

By simplifying, we get,

$$b_n = \frac{A}{n\pi}(-1)^{(n+1} \tag{28}$$

Now we can write the Saw tooth force function as:

$$F(t) = \frac{A}{\pi}\left[Sin\ \omega t - \frac{1}{2}Sin\ 2\omega t + \frac{1}{3}Sin3\omega t - \dots\right] \tag{29}$$

If the damped HarmoniNow we have to solve a series of Newton's equations of motion which looks like:

$$\ddot{x}_n + 2\beta\dot{x}_n + \omega_0^2 x_n = \frac{A}{n\pi}(-1)^{(n+1}Sin\ n\omega t \tag{30}$$

Each one of these equation is no different from the forced Harmonic Oscillator problem we did in class, which takes the form $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_0 Cos\omega t$, for which we know the solution. Which means, we can easily solve for each x_n and then, the total solution can be obtained as $x(t) = \sum_n x_n(t)$

In the website, there is a cdc player, where you can observe the Fourier expansion of Sawtooth function with different components,. Play with it and get an idea. Following shows few snapshots from it.

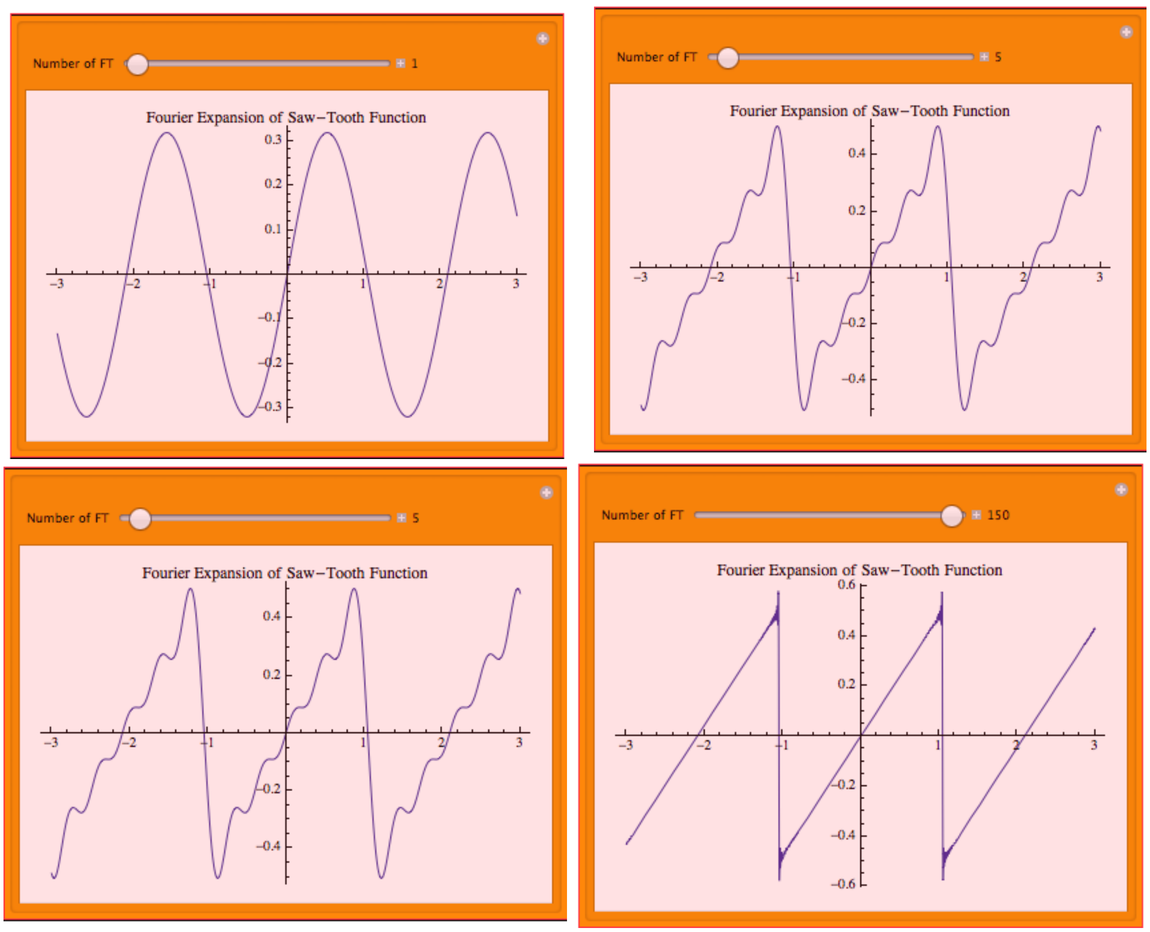


Figure 1: The Fourier Expansion of Sawtooth function. The figure shows one, Two, FIve and 150 Fourier Components in the Expansion, We see the overshooting at the edge, in all orders of expansions, which is known as the Gibbs phenomenon