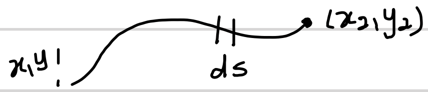
HW#7 Solutions

Q-1 Show that the shortest path between two points on a plane is a straight line.



We can write the length of the line segment as,

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \int \sqrt{1 + \frac{dy^2}{dx^2}}\, dx$$

We can minimize this functional using Euler's equation $f = \sqrt{1 + y'^2(x)}$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = C_1$$

$$\frac{1}{2}\frac{2y'}{\sqrt{1+y'^2}} = C$$

$$\frac{y'^2}{1+y'^2} = C_2 \qquad y'^2 - C_2 y_1'^2 = C_2$$

$$y_1'^2(1 - C_2) = C_2$$

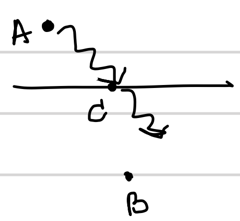
$$y_1' = \sqrt{\frac{C_2}{1 - C_2}} = m$$

$$\frac{dy}{dx} = m$$

$$y = mx + b$$ //

Q-2 Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2. Use the Fermat's principle (light takes the quickest path) to minimize the time and derive the law of refraction $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

Now we use the fact that light has a constant velocity in a given medium. In fact that is how the index of refraction is defined.



$$n = c/v \qquad v = c/n$$

Now let's consider the path of light A to C:

$$\int dt = \int \frac{ds}{v_1} \quad \text{should be minimized}$$

$$\int \frac{\sqrt{dx^2 + dy^2}}{v_1} = \int \frac{\sqrt{1 + y'^2}}{v_1}\, dx \quad \text{should be minimized.}$$

$\int \frac{\sqrt{1+y'^2}}{v_1} dx$ should be minimized, $\quad f = \frac{\sqrt{1+y'^2}}{v_1}$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = b$$

$$\frac{2y'}{2\sqrt{1+y'^2}\, v'} = b$$

$$\frac{y'^2}{(1+y'^2)v'} = b^2 \qquad y' = \frac{dy}{dx} = \tan\theta$$

$$\frac{\tan^2\theta}{1+\tan^2\theta} = bv_1^2$$

$$\frac{\tan^2\theta}{\sec^2\theta} = bv_1^2 \qquad \frac{\sin^2\theta}{\cos^2\theta \cdot \frac{1}{\cos^2\theta}} = bv_1^2$$

$$\sin^2\theta = bv_1^2$$

$$\sin^2\theta = b/(c/v_1)^2 \Rightarrow \sin^2\theta = \frac{b}{n_1^2}$$

$$n_1^2 \sin^2\theta = b$$

$$n \sin\theta = \text{Constant}$$

$\boxed{Q\text{-}3}$ Set up the differential equation to find the shortest path between the (x, y, z) points $(0, -1, 0)$ and $(0\ 1\ 0)$ on the conical surface $z = 1 - \sqrt{x^2+y^2}$. Note that this is the shortest Mountain path around a volcano.

$z = 1 - \sqrt{x^2+y^2} = 1 - r$ (This is a conical surface. So it is easier to do this problem in cylindrical coordinates

We want to find the shortest distance on the cone.

$$\int ds = \int \sqrt{dr^2 + dz^2 + r^2 d\varphi^2} = \int dr^2 + dr^2 + r^2 d\varphi^2 = \int \sqrt{2 + r^2\left(\frac{d\varphi}{dr}\right)^2}\, dr$$

We can use the Euler's Equation with $f = \sqrt{2 + r^2\left(\frac{d\varphi}{dr}\right)^2}$

$$\frac{d}{dr}\frac{\partial f}{\partial \varphi'} - \frac{\partial f}{\partial \varphi} = 0$$

$$\frac{d}{dr}\frac{\partial f}{\partial \varphi'} = 0 \Rightarrow \frac{\partial f}{\partial \varphi'} = \text{Constant.} = \frac{1}{\sqrt{2+r^2\varphi'^2}}\, r^2 2\varphi' = \text{Constant.}$$

$$r^2 4\varphi'^2 = a(2 + r'^2\varphi'^2)$$

You can then solve for $\varphi(r)$.

Q4 Find the curve $y(x)$ that passes through the points $(0,0)$ and $(1,1)$ and minimizes the functional $I[y] = \int_0^1 \left[\left(\frac{dy}{dx}\right)^2 - y^2 \right] dx$.

(a) What is the minimum value of integral?

(b) Evaluate $I[y]$ for a straight line $y=x$ between two points $(0,0)$ & $(1,1)$

Let's first minimize $I[y]$ using the Euler's Equations.

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \qquad f = y'^2 - y^2$$

$$\frac{d}{dx} 2y' - (-2y) = 0$$

$$y'' + y = 0$$

$$y = A\cos(x+\delta)$$

$x=0 \quad y=0 \qquad \delta = \pi/2$

$$y = A \sin x$$

$x=1 \quad y=1 \qquad 1 = A \sin \cancel{1}$ The minimum path $y = \frac{\sin x}{\sin 1}$

$A = 1/\sin 1$

Now we can use this to find the minimum value for $I[y]$

$$y = \frac{\sin x}{\sin 1} \qquad y' = \frac{\cos x}{\sin 1} \qquad y'' = -\frac{\sin x}{\sin 1}$$

$$I = \int_0^1 [y']^2 - y^2 dx = \int_0^1 \frac{\cos^2 x - \sin^2 x}{\sin^2 1} dx = \int_0^1 \frac{\cos 2x \, dx}{\sin^2 1} = \left.\frac{\sin 2x}{2\sin^2 1}\right|_0^1 = \frac{\sin 2}{2\sin^2 1} = 0.64$$

Now if we use $y=x$, definitely $I[y]$ should be larger than 0.64

$$I = \int_0^1 [1]^2 - x^2 dx = \int_0^1 (1-x^2) dx = \left. x - \frac{x^3}{3} \right|_0^1 = 1 - \frac{1}{3} = \frac{2}{3} = 0.67$$

This example in fact demonstrates the fact that Euler's Equation find the condition for minimizing the integral quantity.