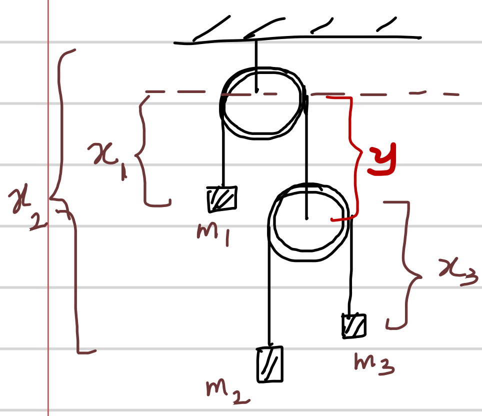
HW 8- Solutions

Q-1) Use the Lagrangean undetermined multiplier method to find the tensions of both strings of the double Atwood machine of example 7-8.



We use 3 coordinates x_1, x_2, x_3 to explain the motion. They are not generalized coordinates. They are related by constraints.

$$T = \frac{1}{2}\left[m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2\right] \quad \text{— (1)}$$

$$V = -mg x_1 - mg x_2 - mg x_3 \quad \text{— (2)}$$

We have constrains

$$x_1 + y = l_1$$

$$x_2 - y + x_3 - y = l_2$$

$$2x_1 + x_2 + x_3 = 2l_1 + l_2$$

$$2\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$$

$$2\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0 \quad \text{— (3)}$$

$$L = T - V$$

$$= \frac{1}{2}\left[m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2\right] + m_1 g x_1 + m_2 g x_2 + m_3 g x_3.$$

Now we apply the Lagrangean Equation with undetermined multipliers. w.r.t. x_1, x_2 and x_3.

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) + \lambda \frac{\partial g}{\partial x_1} = 0 \qquad m g - m_1 \ddot{x}_1 + \lambda 2 = 0 \quad \text{— (4)}$$

$$\frac{\partial L}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) + \lambda \frac{\partial L}{\partial x_2} = 0 \qquad m_2 g - m_2 \ddot{x}_2 + \lambda = 0 \quad \text{— (5)}$$

$$\frac{\partial L}{\partial x_3} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) + \lambda \frac{\partial L}{\partial x_3} = 0 \qquad m_3 g - m_3 \ddot{x}_3 + \lambda = 0 \quad \text{— (6)}$$

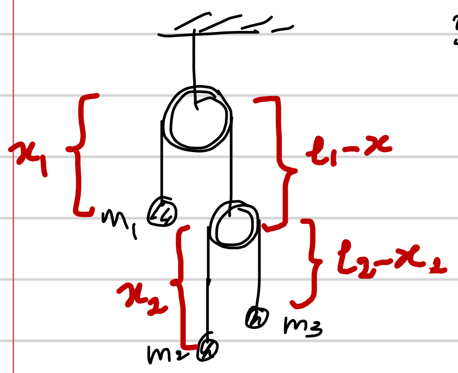
$$\frac{(4)}{m_1} \times 2 + \frac{(5)}{m_2} + \frac{(6)}{m_3} \qquad 4g - 2\ddot{x}_1 - \ddot{x}_2 - \ddot{x}_3 + \frac{4\lambda}{m_1} + \frac{\lambda}{m_2} + \frac{\lambda}{m_3} = 0$$

$$4g + \lambda\left[\frac{4}{m} + \frac{1}{m_2} + \frac{1}{m_3}\right] = 0$$

$$\lambda = \frac{-4g}{4/m_1 + 1/m_2 + 1/m_3}$$

We can calculate the generalized force $T_1 = \lambda \frac{\partial g}{\partial x_1} = \lambda 2 = \frac{-8g}{4/m_1 + 1/m_2 + 1/m_3}$ and so on.

Q-2) Determine the Hamiltonian and Hamilton's Equations of motion for the double Atwood machine of example 7.8.



y-coordinate for $m_1 \rightarrow x_1$

" $m_2 = l_1 - x_1 + x_2$

$m_3 = l_1 - x_1 + l_2 - x_2$

$\dot{y}_{m_1} = \dot{x}_1$

$\dot{y}_{m_2} = -\dot{x}_1 + \dot{x}_2$

$\dot{y}_{m_3} = -\dot{x}_1 - \dot{x}_2$

We need two generalized coordinates to explain the motion.

$$T = \frac{1}{2} m_1 [\dot{x}_1^2] + \frac{1}{2} m_2 [-\dot{x}_1 + \dot{x}_2]^2 + \frac{1}{2} m_3 [-\dot{x}_1 - \dot{x}_2]^2$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} m_3 (\dot{x}_1^2 + \dot{x}_2^2) + \frac{2 \dot{x}_1 \dot{x}_2 (m_3 - m_2)}{2}$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} (m_2 + m_3)(\dot{x}_1^2 + \dot{x}_2^2) + \dot{x}_1 \dot{x}_2 (m_3 - m_2)$$

$$V = -m_1 g x_1 - m_2 g (l_1 - x_1 + x_2) - m_3 g (l_1 - x_1 + l_2 - x_2)$$

$$= -(m_1 + m_2 + m_3) g x_1 - (m_2 + m_3) g x_3 - m_2 g l_1 - m_3 g (l_1 + l_2)$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2}(m_2 + m_3)(\dot{x}_1^2 + \dot{x}_2^2) + \dot{x}_1 \dot{x}_2 (m_3 - m_2) + (m_1 + m_2 + m_3) g x_1 - (m_2 + m_3) g x_3$$

(We have dropped the constant energy term here)

Now let's define the generalized momenta.

$$P_{x_1} = \frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 + (m_2 + m_3)\dot{x}_1 + \dot{x}_2 (m_3 - m_2) = (m_1 + m_2 + m_3)\dot{x}_1 + (m_3 - m_2)\dot{x}_2$$

$$P_{x_2} = \frac{\partial L}{\partial \dot{x}_2} = (m_2 + m_3)\dot{x}_2 + (m_3 - m_2)\dot{x}_1$$

$$P_{x_1} = (m_1 + m_2 + m_3)\dot{x}_1 + (m_3 - m_2)\dot{x}_2 \quad \text{—— (1)}$$

$$P_{x_2} = (m_3 + m_2)\dot{x}_2 + (m_3 - m_2)\dot{x}_1 \quad \text{—— (2)}$$

First what we want is to write $\dot{x}_1$ & $\dot{x}_2$ interms of P_{x_1} & P_{x_2}.

(1) $\times (m_3 + m_2)$ − (2) $\times (m_3 - m_2)$

$$P_{x_1}(m_3 + m_2) - P_{x_2}(m_3 - m_2) = \left[(m_3 + m_2)(m_1 + m_2 + m_3) - (m_3 - m_2)^2\right] \dot{x}_1$$

$$= \left[m_1 m_3 + m_2 m_3 + \cancel{m_3^2} + m_1 m_2 + \cancel{m_2^2} + m_3 m_2 - \cancel{m_3^2} + 2 m_3 m_2 - \cancel{m_2^2}\right] \dot{x}_1$$

$$= \left[m_1 m_3 + m_2 m_3 + m_1 m_2 + m_2 m_3 + 2 m_2 m_3\right] \dot{x}_1$$

$$= \left[m_1 m_3 + m_1 m_2 + 4 m_2 m_3\right] \dot{x}_1$$

$$\dot{x}_1 = \frac{(m_3+m_2)\,P_{x_1} - (m_3-m_2)\,P_{x_2}}{(m_1m_3 + m_1m_2 + 4m_2m_3)} \quad \text{——} (3)$$

Now we can substitute (3) in (2)

$$P_{x_2} = (m_3+m_2)\,\dot{x}_2 + (m_3-m_2)\,\dot{x}_1$$

$$= (m_3-m_2)\,\frac{[(m_3+m_2)P_{x_1} - (m_3-m_2)P_{x_2}}{m_1m_3+m_1m_2+4m_2m_3} + (m_3+m_2)\dot{x}_2$$

$$P_{x_2}(m_1m_3+m_1m_2+4m_2m_3) - (m_3+m_2)\left[(m_3+m_2)P_{x_1} + (m_3-m_2)P_{x_2}\right) = (m_1m_3+m_1m_2+4m_2m_3)(m_3+m_2)\dot{x}_2$$

$$P_x^2\left[m_1m_3+m_1m_2+4m_2m_3+(m_3-m_2)^2\right]P_{x_2} + \left[m_2^2-m_3^2\right]P_{x_1} = \text{"}$$

$$P_{x_2}\left[m_1m_3+m_1m_2+4m_2m_3+m_3^2+m_2^2-2m_2m_3\right] + \left[(m_2-m_3)(m_2+m_3)P_{x_1}\right] = \text{"}$$

$$P_{x_1}\left[m_1m_3+m_1m_2+(m_2+m_3)^2 + \left[(m_2-m_2)(m_2+m_3)\right]\right]P_{x_1} = \text{"}$$

$$P_{x_1}(m_2+m_3)\left[m_1+m_2+m_3\right] + m_2+m_3\left[(m_2-m_3)P_{x_1}\right] = (m_2+m_3)(m_1m_2+m_1m_3+4m_2m_3)\,\dot{x}_2$$

$$P_{x_1}\left[m_1+m_2+m_3\right] + P_{x_2}(m_2-m_3)$$

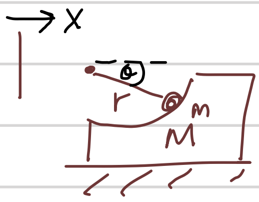
(Q3) A particle of mass m slides down a smooth circular wedge of mass M as shown in the figure. The wedge rests on a smooth horizontal table. Find

(a) The equation of motion for m & M

(b) the reaction of the wedge on m.

$x_M = x \qquad x_m = x + r\cos\theta \qquad \dot{x}_m = \dot{x} + \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$y_M = 0 \qquad y_m = -r\sin\theta \qquad \dot{y}_m = -\dot{r}\sin\theta - r\dot{\theta}\cos\theta$



$$T_M = \frac{1}{2}M\dot{x}^2$$

$$T_m = \frac{1}{2}m\left[(\dot{x}+\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (-\dot{r}\sin\theta - r\dot{\theta}\cos\theta)^2\right]$$

$$= \frac{1}{2}m\left[\dot{x}^2 + 2\dot{x}(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) + \dot{r}^2 + r^2\dot{\theta}^2\right]$$

$$T_{total} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta} + 2\dot{x}(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\right]$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{x}(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\right] + mgr\sin\theta$$

Constraint $g(r) = r - R = 0$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \lambda\frac{\partial g}{\partial x} = 0 \Rightarrow \frac{d}{dt}\left[(M+m)\dot{x} - m\ r\dot{\theta}\sin\theta\right] = 0$$

$$(M+m)\ddot{x} - m\dot{r}\cancel{\dot{\theta}\sin\theta} - mr\dot{\theta}\dot{\theta}\cos\theta - mr\ddot{\theta}\sin\theta = 0$$

$$(M+m)\ddot{x} = mr(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

$$\ddot{x} = \frac{mR}{M+m}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) + \lambda\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

We can now apply $\dot{r} = 0$ $\quad L = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}\left[r^2\dot{\theta}^2 - 2\dot{x}r\dot{\theta}\sin\theta\right] + mgr\sin\theta$

$$-m\dot{x}r\dot{\theta}\cos\theta + mgr\cos\theta - \frac{d}{dt}\left[mr^2\dot{\theta} - m\dot{x}r\sin\theta\right]$$

$$-m\dot{x}\cancel{r\dot{\theta}}\cos\theta + mgr\cos\theta + mR^2\ddot{\theta} - m\ddot{x}R\sin\theta + m\dot{x}\cancel{r\dot{\theta}}\cos\theta = 0$$

$$MR^2\ddot{\theta} + mgR\cos\theta - m\ddot{x}R\sin\theta = 0$$

$$\ddot{\theta} \cdot \frac{\ddot{x}\sin\theta - g\cos\theta}{R}$$

w.r.t. $\quad L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{x}\dot{r}\cos\theta - 2\dot{x}r\dot{\theta}\sin\theta\right) + mgr\sin\theta$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) + \lambda\frac{\partial g}{\partial r} = 0$$

$$mr\dot{\theta}^2 - m\dot{x}\dot{\theta}\sin\theta + mg\sin\theta - \frac{d}{dt}\left[m\dot{r} + m\dot{x}\cos\theta\right] + \lambda = 0$$

$$mR\dot{\theta}^2 - m\dot{x}\cancel{\dot{\theta}\sin\theta} + mg\cos\theta - m\ddot{x}\sin\theta + m\cancel{\dot{x}\dot{\theta}}\sin\theta + \lambda = 0$$

$$\lambda = m\ddot{x}\sin\theta - mR\dot{\theta}^2 - mg\sin\theta$$

Q4 A particle of m is attracted to force with the force of magnitude k/r^2. Use the plane polar coordinate. Find the Hamiltonian equation of motion.

$$T = \frac{1}{2} m r^2 + \frac{1}{2} m r^2 \dot{\theta}^2 \qquad V = -k/r$$

$$L = \frac{1}{2} m \dot{r}^2 + r^2 \dot{\theta}^2 + \frac{k}{r}$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \implies \dot{r} = P_r/m$$

$$P_\theta = mr^2\dot{\theta} \qquad \theta = P_\theta / mr^2$$

$$H = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2 \dot{\theta}^2 - \frac{k}{r}$$

$$= \frac{1}{2} m \left(\frac{P_r}{m}\right)^2 + \frac{1}{2} mr^2 \left(\frac{P_\theta}{mr^2}\right)^2 - k/r$$

$$= \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} - \frac{k}{r}$$

Hamiltonian Equations:

$$\dot{r} = \frac{\partial H}{\partial P_r} = \frac{P_r}{m} \qquad \dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mr^2}$$

$$\dot{P}_r = -\frac{\partial H}{\partial r} = \frac{P_\theta^2}{mr^2} - \frac{k}{r^2}$$

$$\dot{P}_\theta = 0 \qquad P_\theta = \text{Constant}.$$

Q5) The potential from anharmonic oscillator is $U = \frac{kx^2}{2} + \frac{bx^4}{4}$, where k and b constants.
Find the Hamilton's Equations of Motions.

$$T = \frac{1}{2} m\dot{x}^2 \qquad U = \frac{1}{2} kx^2 + \frac{1}{4} bx^4$$

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 - \frac{1}{4} bx^4$$

$$H = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 + \frac{1}{4} bx^4$$

$$= \frac{P_x^2}{2m} + \frac{1}{2} kx^2 + \frac{1}{4} bx^4$$

$$\dot{x} = \partial H / \partial P \qquad \dot{x} = P_x / m$$

$$\dot{p} = \frac{-\partial H}{\partial x} = -\left[kx + bx^3 \right] \Rightarrow \dot{P}_x = -\left[+kx + bx^3 \right]$$