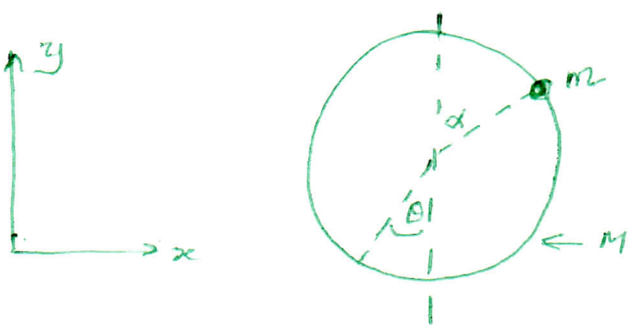
problem 1 A point particle of mass m is constrained to move frictionlessly on the inside of a circular wire hoop of radius r, ~~and~~ uniform density and mass M. The hoop is in the x-y plane, can roll on a fixed line in the x-axis, but does not slide, nor does it loose the contact with the x-axis. The point particle is acted on by gravity exerting force along the negative y-axis. At $t=0$ suppose the hoop is at rest. At this time the particle is at the top of the hoop, and is given a velocity v_0 along the x-axis. What is the velocity v_f with respect to the fixed axis, when the particle comes to the bottom of the hoop? Simplify your answer in the limits $m/M \to 0$ and $M/m \to 0$



Let's use the fixed coordinate system as shown

$x_M = x + r \sin\alpha$ $\qquad$ $\dot{x}_m = \dot{x} + \cancel{\dot{r}\sin\alpha} + r\dot{\alpha}\cos\alpha$

$y_M = r + r\cos\alpha$ $\qquad$ $\dot{y}_m = \cancel{\dot{x}} + \cancel{\dot{r}\cos\alpha} \mp r\dot{\alpha}\sin\alpha$

$\dot{x}_m = \dot{x} + r\dot{\alpha}\cos\alpha$

$\dot{y}_m = -r\dot{\alpha}\sin\alpha$

$$T_m = \frac{1}{2} m \left[(\dot{x} + r\dot{\alpha}\cos\alpha)^2 + (-r\dot{\alpha}\sin\alpha)^2 \right]$$

$$T_m = \frac{1}{2} m \left[\dot{x}^2 + r^2\dot{\alpha}^2 + 2\dot{x}r\dot{\alpha}\cos\alpha \right] \text{ — ①}$$

For the hoop

$$X_M = x$$

$$T_M = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} I\dot{\theta}^2 \qquad I = Mr^2$$

Because hoop is not sliding

$$T_M = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} Mr^2 \left[\frac{\dot{x}}{r}\right]^2 \qquad x = r\theta, \quad \theta = \frac{x}{r}$$

$$T_M = M\dot{x}^2 \text{ — ②}$$

Potential Energy

$$U_M = \text{Const.}$$

$$U_m = mg(r + r\cos\alpha) \text{ — ③}$$

①, ②, & ③

$$L = M\dot{x}^2 + \frac{1}{2} m \left[\dot{x}^2 + r^2\dot{\alpha}^2 + 2\dot{x}r\dot{\alpha}\cos\alpha \right] - mg(r + r\cos\alpha) \text{ — ④}$$

[NOTE: We use x & α as generalized coordinates]

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial \dot{x}} = \text{Constant} = b$$

$$2M\dot{x} + m\dot{x} + mr\dot{\alpha}\cos\alpha = b \text{ — ⑤}$$

We can find the initial conditions to find what b is.

At the first place, in eq (A) we wrote the velocity of the bead.

$$\left.\begin{aligned}\dot{x}_m &= \dot{x} + r\dot{\alpha}\cos\alpha \\ \dot{y}_m &= -r\dot{\alpha}\sin\alpha.\end{aligned}\right\} \text{—— (A)}$$

At $t=0$ ~~particle~~ hoop is at rest. Particle is given a velocity V_0 along $\dot{x}$

that gives

from (A) $\dot{x}_m = r\dot{\alpha} = V_0$ —— (B)

~~go back to equatio~~

go back to eq^n (5) $m r\dot{\alpha}\Big|_{at\ t=0} = b$

$$m V_0 = b.$$

Eq^n (5) becomes

$$2M\dot{x} + m\dot{x} + m r\dot{\alpha}\cos\alpha = mV_0 \text{ —— (6)}$$

We want to find beads velocity at $\theta = \pi$

from (A)

at $\theta = \pi$ $\quad \dot{x}_m = \dot{x} + r\dot{\alpha}(-1) = \dot{x} - r\dot{\alpha}$

$$\dot{y}_m = 0$$

from eq^n (6) $\quad 2M\dot{x} + m\dot{x} + m r\dot{\alpha}(-1) = mV_0$

$$2M\dot{x} + m(\dot{x} - r\dot{\alpha}) = mV_0$$

(C) —— $2M\dot{x} + mV_f = mV_0$

If we have $\dot{x}$ (at $\theta = \pi$) we know V_f.

For that we can use conservation of energy to find $\dot{x}$ at $\theta = \pi$

$$\underbrace{M\dot{x}^2 + \frac{1}{2} m V_f^2}_{\text{total energy at } \theta = \pi} = \underbrace{\frac{1}{2} m V_0^2 + 2mgr}_{\text{total energy at } t=0,\ \theta=0} \quad \text{—— (D)}$$

By (C) & (D), we can find V_f and apply limits.

~~NOTE:~~

Problem (2) Consider a particle of mass m, which is constrained to move on the surface of a sphere of radius R. There are no exteternal forces of any kind on this particle.

(a) What are the generalized coordinates necessary to describe the problem?

(b) What is the Hamiltonian of the System. Is the Hamiltonian conserved:

(c) Set up the Hamiltonian equations of motion.

(r,θ,φ) ~~explai~~ can be used to explain the system with the constraint $r = R$

$(\theta,\varphi) \longrightarrow$ Generalized Coordinates.

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2 \right]$$

[NOTE : Here I have used the ~~coordinates of~~ v^2 for spherical coordinates]

$r = R$

$\dot{r} = 0$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta \right)$$

$V = 0$ [No forces of any kind]

$$L = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta \right) \quad \text{—— (1)}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta} \quad \text{—— (2)}$$

$$P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m R^2 \dot{\varphi} \sin^2\theta \quad \text{—— (3)}$$

Now we write the Hamiltonian. Since the co-ordinate transformation has no ~~time~~ explicit time dependance

$$H = T + V$$

$$= \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \dot{\varphi}^2 \sin^2\theta$$

$$= \frac{mR^2}{2}\left(\frac{P_\theta}{mR^2}\right)^2 + \frac{1}{2} mR^2 \left(\frac{P_\varphi}{mR^2 \sin^2\theta}\right)^2 \sin^2\theta$$

$$H = \frac{P_\theta^2}{2mR^2} + \frac{P_\varphi^2}{2mR^2\sin^2\theta} \quad \text{—— (4)}$$

~~H~~ ~~dH/dt~~ Lagrangean does not have an explicit time dependance. Hamiltonian (in this case Energy) is conserved.

(c) Hamilton's Equations of Motion.

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{2P_\theta}{2mR^2} \qquad P_\theta = mR^2\dot{\theta}$$

$$-\dot{P_\theta} = \frac{\partial H}{\partial \theta} = \frac{P_\varphi^2}{2mR^2}\,\frac{(-2)}{\sin^3\theta}\cos\theta \qquad \dot{P_\theta} = \frac{P_\varphi^2 \cos\theta}{2mR^2\sin^3\theta}$$

$$\dot{\varphi} = \frac{\partial H}{\partial P_\varphi} = \frac{2P_\varphi}{2mR^2\sin^2\theta} \qquad P_\varphi = mR^2\sin^2\theta\,\dot{\varphi}$$

$$-\dot{P_\varphi} = \frac{\partial H}{\partial \varphi} = 0 \qquad P_\varphi = \text{Constant.}$$

Problem (3) A particle under the action of gravity slides on the inside of a paraboloid of revolution whose axis is vertical. (Suppose that the paraboloid of revolution is generated by a parabola which is defined by $z = Ar^2$ in cylindrical coordinates (r, ϕ, z). Using the distance from the axis r, and the azimuthal angle ϕ as generalized coordinates, Find

(a) The Lagrangean of the system.

(b) The generalized momenta & the corresponding Hamitonian.

(c) The equation of motion.

(d) If $\dot{\phi} = 0$, show that the particle can execute small oscilliations about the lowest point of the paraboloid and find the frequency of Oscillations.

We can use (r, z, φ) ~~as generalized coordinates~~ to explain the motion

$$z = Ar^2 \longrightarrow \text{Constraint.}$$

So $(r, \varphi) \longrightarrow$ Generalized Coordinates

$$L = T - V = \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{z}^2 \right] - mgz$$

$$= \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\varphi}^2 + (2Ar\dot{r})^2 \right] - mgz$$

$$= \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\varphi}^2 + 4A^2 r^2 \dot{r}^2 \right] - mgz$$

$$= \frac{1}{2} m (1 + 4A^2 r^2) \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 - mgz \quad \text{--- (1)}$$

(b) The generalized momenta are:

$$p_r = \frac{\partial L}{\partial \dot r} = m(1+4A^2r^2)\dot r$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = mr^2\dot\varphi$$

(c) $H = T + V$

$$= \frac{1}{2} m(1+4A^2r^2)\dot r^2 + \frac{1}{2} mr^2\dot\varphi^2 - mgz$$

$$= \frac{1}{2} m(1+4A^2r^2)\left(\frac{p_r}{(1+4A^2r^2)m}\right)^2 + \frac{1}{2} mr^2\left(\frac{p_\varphi}{mr^2}\right)^2 - mgz$$

$$H = \frac{p_r^2}{2m(1+4A^2r^2)} + \frac{p_\varphi^2}{2mr^2} - mgz$$

(e) Equation of Motion (You can use either the Lagrangean ~~gean~~ or Hamilton's eq^n of motion)

$$\dot r = \frac{\partial H}{\partial p_r} \qquad \dot r = \frac{2p_r}{2m(1+4A^2r^2)} \qquad p_r = m(1+4A^2r^2)\dot r$$

$$\dot\varphi = \frac{\partial H}{\partial p_\varphi} \qquad \dot\varphi = \frac{2p_\varphi}{2mr^2} \qquad p_\varphi = mr^2\dot\varphi$$

$$-\dot p_r = \frac{\partial H}{\partial r} = \frac{p_r^2}{2m(1+4A^2r^2)^2}\,(-1)\,8A^2r + \frac{p_\varphi^2}{2mr^3}(-2)(2r).$$

$$+\dot p_r = \frac{p_r^2\,4A^2r}{(1+4A^2r^2)^2} + \frac{2p_\varphi^2\,r}{2mr^3}$$

$$\frac{d}{dt}\, m(1+4A^2r^2)\dot r = \frac{p_r^2\,4A^2r}{m(1+4A^2r^2)^2} + \frac{2p_\varphi^2}{2mr^2}$$

$$H = \frac{P_r^2}{2m(1+4A^2r^2)} + \frac{P_\varphi}{2mr^2} - mgAr^2$$

(c) Hamilton's Equations.

$$\dot{r} = \frac{\partial H}{\partial P_r} = \frac{2P_r}{2m(1+4A^2r^2)} \qquad P_r = m(1+4A^2r^2)\dot{r} \quad \text{—— (1)}$$

$$\dot{\varphi} = \frac{\partial H}{\partial P_\varphi} = \frac{2P_\varphi}{2mr^2} \qquad P_\varphi = mr^2\dot{\varphi}^2 \quad \text{—— (2)}$$

$$-\dot{P}_r = \frac{\partial H}{\partial r} = \frac{P_r^2}{2m}\frac{(-1)\,8A^2r}{(1+4A^2r^2)^2} + \frac{P_\varphi^2}{2m}\frac{(-2)(2r)}{r^3} - mgA2r$$

$$\dot{P}_r = \frac{P_r^2\, 8A^2r}{2m(1+4A^2r^2)^2} + \frac{4P_\varphi^2 r}{2mr^3} - 2mgAr$$

$$\dot{P}_r = m(1+4A^2r^2)\ddot{r} + m\dot{r}\,8A^2r\dot{r}$$

$$m(1+4A^2r^2)\ddot{r} + 8A^2mr\dot{r}^2 = \frac{4A^2rP_r^2}{m(1+4A^2r^2)^2} + \frac{2P_\varphi^2 r}{mr^3} - 2mgAr$$

$$= \frac{4A^2r[m\cancel{(1+4A^2r^2)}\dot{r}]^2}{m\cancel{(1+4A^2r^2)^2}} + \frac{2[mr^2\dot{\varphi}]^2 r}{mr^3} - 2mgAr$$

$$= 4A^2rm\dot{r}^2 + 2mr\dot{\varphi}^2 - 2mgAr$$

$$m(1+4A^2r^2)\ddot{r} + 4A^2mr\dot{r}^2 - 2mr^2\dot{\varphi}^2 + 2mgAr = 0 \quad \text{—— (3)}$$

$$-\dot{P}_\varphi = \frac{\partial H}{\partial \varphi} = 0 \qquad P_\varphi = \text{Const.}$$

$$mr^2\dot{\varphi} = \text{const} \quad \text{—— (4)}$$

(d) If $\dot{\varphi} = 0$

Eq^{n} (3) becomes

$$m(1+4A^2r^2)\ddot{r} + 4A^2 m r\dot{r}^2 + 2mgAr = 0$$

For small oscilliations $r \to 0$

$\dot{r} \to 0$

$\ddot{r} \to 0$

Taking only the 1st order derms

$$m\ddot{r} + 2mgAr = 0$$

$$\ddot{r} + 2gAr = 0$$

$$\ddot{r} + \omega^2 r = 0$$

$$\omega = \sqrt{2gA}$$