PHYS 310 - HW # 2

Q-1 Suppose that the force acting on a particle is factorable in the following form.

(a) $F(x_i, t) = f(x_i)\, g(t)$

(b) $F(\dot{x}_i, t) = f(\dot{x}_i)\, g(t)$

(c) $F(x_i, \dot{x}_i) = f(x_i)\, g(\dot{x}_i)$

For which cases are these equations of motion integrable?

(a) $F(x_i, t) = f(x_i)\, g(t)$

$$F = m\ddot{x}$$

$$f(x_i)\, g(t) = m\ddot{x}_i = m\frac{d\dot{x}_i}{dt}$$

$$g(t)\, dt = \frac{m}{f(x_i)}\, d\dot{x}_i \quad \rightarrow \text{Non integrable}$$

(b) $F = f(\dot{x}_i)\, g(t)$

$$m\ddot{x}_i = f(\dot{x}_i)\, g(t)$$

$$m\frac{d^2 x_i}{dt^2} = f(\dot{x}_i)\, g(t)$$

$$\frac{m}{f(\dot{x}_i)}\, d\dot{x}_i = g(t)\, dt$$

$\rightarrow$ integrable

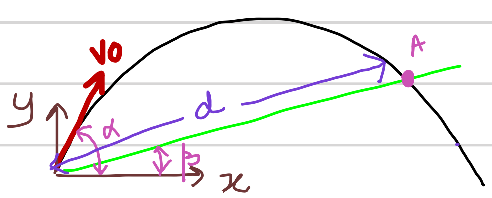
(c) $F = f(x_i)\, g(\dot{x}_i)$

$$m\ddot{x}_i = f(x_i)\, g(\dot{x}_i)$$

$\rightarrow$ Non integrable

20 points

Q-2 If a projectile is fired from the origin of the coordinate system with an initial velocity v_0, and in a direction making an angle α with the horizontal, calculate the time required for the projectile to cross a line passing through the origin and making an angle $\beta \leq \alpha$ with the horizontal.



We need to know the time when the projectile is at A in the diagram.

Basically, we need to consider the motion of the projectile from the origin O to point A.

The only force acting on the particle is the gravitational force i.e. $F_x = 0$

$F_y = -mg$

Now let's apply the Newton's equations in the $\hat{x}$ and $\hat{y}$ directions.

$$F_x = m\ddot{x} \quad \& \quad F_y = m\ddot{y}$$

$\hat{x}$ direction:

$$F_x = m\ddot{x}$$
$$0 = m\ddot{x}$$
$$\dot{x} = \text{Constant}$$
$$\dot{x} = C \quad \text{—— (1)}$$

This equation basically says that the x component of the velocity of the projectile is a constant through out the motion.

(Which makes sense as there is no force in the $\hat{x}$ direction)

Using the boundary conditions in eq (1)

at $t=0$ $V_x = V_0 \cos\alpha$

$$\dot{x} = V_0 \cos\alpha$$

In the $\hat{y}$ direction, $-mg = m\ddot{y}$

$$-g = \ddot{y}$$

Integrating:

$$-gt = \dot{y} + C_1$$

intial conditions $t=0$ $\dot{y} = V_0 \sin\alpha$

$$C_1 = -V_0 \sin\alpha$$

$$\dot{y} = -gt + V_0 \sin\alpha$$

Now the two equations we have:

$$\dot{x}(t) = V_0 \cos\alpha \quad \text{—— (2)}$$
$$\dot{y}(t) = -gt + V_0 \sin\alpha \quad \text{—— (3)}$$

By integrating (2) & (3) and then using the initial conditions,

$t=0$ $x=0$, $y=0$

$$x(t) = V_0 \cos\alpha\, t$$
$$y(t) = -\frac{g t^2}{2} + V_0 \sin\alpha\, t$$

We are looking for the condition $x(t) = d\cos\beta$ and $y(t) = d\sin\beta$

$$d\cos\beta = V_0 \cos\alpha\, t_0 \quad \text{—— (4)}$$
$$d\sin\beta = -\frac{g t_0^2}{2} + V_0 \sin\alpha\, t_0 \quad \text{—— (5)}$$

$$(5)/(4) \quad \tan\beta = \frac{-g t_0^2/2 + V_0 \sin\alpha\, t_0}{V_0 \cos\alpha\, t_0}$$

$$t_0 V_0 \cos\alpha \tan\beta = -\frac{g t_0^2}{2} + V_0 \sin\alpha\, t_0$$

$$t_0 \left[+\frac{g t_0}{2} + V_0 \cos\alpha \tan\beta - V_0 \sin\alpha \right] = 0$$

$t_0 = 0$ or $\frac{g t_0}{2} + V_0 \cos\alpha \tan\beta - V_0 \sin\alpha = 0$

$$t_0 = \frac{2V_0}{g} \left[\sin\alpha - \cos\alpha \tan\beta \right]$$

Q-3 A projectile is fired with a velocity V_0 such that it passes through two points, both a distance h above the horizontal. Show that if the gun is adjusted for maximum range, the seperation of the two points is $d = \frac{V_0}{g}\sqrt{V_0^2 - 4gh}$.

For this problem, (as we did in the problem # 2) we can write

$$\left.\begin{aligned} x(t) &= V_0 t \cos\alpha \\ y(t) &= -\frac{g t^2}{2} + V_0 t \sin\alpha \end{aligned}\right\} \text{—— (1)}$$

Gun is adjusted for the maximum range. For finding the range, we make $y(t) = 0$

$$\frac{g t^2}{2} = V_0 t \sin\alpha$$

$$t = T = \frac{2 V_0 \sin\alpha}{g}$$

→ by putting this $t = T$ in $x(t)$

$$x(t=T) = \frac{2V_0^2}{g} \sin\alpha \cos\alpha = \frac{V_0^2}{g} \sin 2\alpha$$

$x(t)$ becomes max. at $2\alpha = \pi/2$

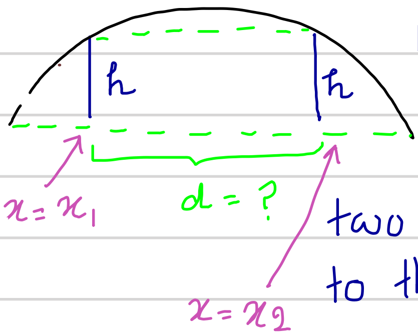
$\alpha = \pi/4$ ⟶

$\alpha = \pi/4$

When the gun is set at $\alpha = \pi/4$, equation 1 reduces to :

$$x(t) = \frac{V_0}{\sqrt{2}} t$$

$$y(t) = \frac{V_0}{\sqrt{2}} t - \frac{1}{2} g t^2$$



We need to find d.
We can find the time when $y(t) = h$
Obviously there are two possibilities according to the diagram.

$$h = \frac{V_0 t}{\sqrt{2}} - \frac{g t^2}{2}$$

$$\frac{g t^2}{2} - \frac{V_0}{\sqrt{2}} t + h = 0$$

$$t^2 - \frac{\sqrt{2} V_0}{g} t + \frac{2h}{g} = 0$$

$$t = \frac{\frac{\sqrt{2} V_0}{g} \pm \sqrt{\frac{2 V_0^2}{g^2} - 4 \frac{2h}{g}}}{2}$$

$$= \frac{V_0}{\sqrt{2} g} \pm \sqrt{\frac{V_0^2}{2 g^2} - \frac{2h}{g}} = \frac{V_0}{\sqrt{2} g} \pm \frac{V_0}{\sqrt{2} g} \left[1 - \frac{4 g h}{V_0^2} \right]^{1/2}$$

Let's see the x coordinates at these possibilities for t :

$$x(t) = \frac{V_0}{\sqrt{2}} t \qquad x_1 = \frac{V_0}{\sqrt{2}} \left[\frac{V_0}{\sqrt{2} g} - \frac{V_0}{\sqrt{2} g} \sqrt{1 - \frac{4 g h}{V_0^2}} \right] \quad \text{—— (2)}$$

$$x_2 = \frac{V_0}{\sqrt{2}} \left[\frac{V_0}{\sqrt{2} g} + \frac{V_0}{\sqrt{2} g} \sqrt{1 - \frac{4 g h}{V_0^2}} \right] \quad \text{—— (3)}$$

(3) − (2)

$$x_2 - x_1 = \frac{V_0^2}{2g} \, 2 \sqrt{1 - \frac{4 g h}{V_0^2}}$$

$$x_2 - x_1 = \frac{V_0^2}{g} \sqrt{1 - \frac{4 g h}{V_0^2}} = \frac{V_0}{g} \sqrt{V_0^2 - 4 g h}$$

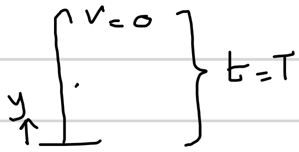
$$x_2 - x_1 = d = \frac{V_0}{g} \sqrt{V_0^2 - 4 g h}$$

$\boxed{Q4}$ Consider a projectile fired vertically in a constant gravitational field. For the same initial velocities compare the time required for the projectile to reach the maximum height.

(a) for zero resistive force

(b) for a resisting force proportional to the instantaneous velocity of the projectile.

(a) Zero resistive force:



The equation of motion $F = m\ddot{y} = -mg$

$$\dot{y} = -gt + c$$

at $t=0$ $\dot{y} = v_0$ $\qquad \dot{y} = -gt + v_0$

Now we look for the condition $\dot{y} = 0$ (That is when it reaches the maximum height.

$$v_0 = gT \qquad T = v_0/g \,//$$

(b) With a resistive force $-kmv$

$$m\ddot{y} = -mg - mkv$$

$$\ddot{y} = -g - k\dot{y}$$

$$\frac{dv}{dt} = -(g + kv)$$

$$\frac{1}{g+kv}\, dv = -g\, dt \Rightarrow \frac{1}{k} \ln(g+kv) = -t + C_2$$

$$C_2 = \frac{1}{k} \ln(g + kv_0)$$

at $t=0$ $v = v_0$

$$\frac{1}{k} \ln\left[\frac{g+kv}{g+kv_0}\right] = -t$$

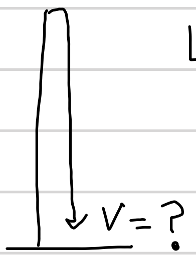
$$\frac{g+kv}{g+kv_0} = e^{-kt} \qquad v = \frac{-g}{k} + \frac{g+kv_0}{k} e^{-kt}$$

we are looking for the condition $v = 0$

$$\frac{g}{k} = \frac{g+kv_0}{k} e^{-kT} \qquad kT = \ln\left[\frac{g+kv_0}{g}\right] \qquad T = \frac{1}{k} \ln\left[1 + \frac{kv_0}{g}\right] \,//$$

Q-5 A particle is projected vertically upward in a constant gravitational field, with an initial speed v_0. Show that, if there is a retarding force proportional to the square of the instantaneous speed, the speed of the particle when it return to the original position is:

$$\frac{v_0 v_t}{\sqrt{v_0^2+v_t^2}}$$ where v_t is the terminal speed.



Let's first work on the upward motion:

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = V\frac{dV}{dx}$$

$$m v \frac{dv}{\cancel{dx}\,dy} = -mg - mkv^2$$

$$\frac{v}{g+kv^2}\,dV = -dy$$

$$\frac{1}{2k}\ln[g+kv^2] = -y + C$$

at $y=0$ $V=V_0$

$$\frac{1}{2k}\ln\frac{g+kv^2}{g+kv_0^2} = -y \Rightarrow y = \frac{1}{2k}\ln\left[\frac{g+kv_0^2}{g+kV^2}\right] \quad \text{—— (1)}$$

From eq. (1). we relate the position and velocity.

$V=0$ at the maximum reach.

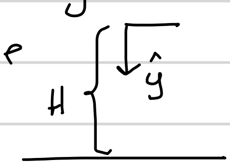
$$H = \frac{1}{2k}\ln\left[\frac{g+kv_0^2}{g}\right] \quad \text{—— (2)}$$

Now when the velocity of the particle becomes zero, gravity will pull it down.

Let's apply the Newton's equation for the downward motion:

$$m\ddot{y} = mg - mkv^2$$

Now we use a coordinate system like



$$\frac{dV}{dt} = g - kv^2$$

$$\frac{VdV}{dy} = g - kv^2$$

$$\frac{V}{g-kv^2}\,dv = g\,dy$$

$$-\frac{1}{2k}\ln[g-kv^2] = gy + C_2$$

at $y=0$ $V=0$ $\quad C_2 = -\frac{1}{2k}\ln g$

$$y = \frac{1}{2k} \ln\left[\frac{g}{g - kv^2}\right].$$

We need to find the velocity when $y = H$

$$H = \frac{1}{2k} \ln \frac{g}{g - kv^2} \quad \text{--- (3)}$$

Now combining equations (2) & (3) $\Rightarrow$

$$\frac{1}{2k} \ln \frac{g + kV_0^2}{g} = \frac{1}{2k} \ln \frac{g}{g - kv^2}$$

$$\frac{g + kV_0^2}{g} = \frac{g}{g - kv^2}$$

$$g - kv^2 = \frac{g^2}{g + kV_0^2}$$

$$kv^2 = g - \frac{g^2}{g + kV_0^2}$$

$$v^2 = \frac{g}{k} - \frac{g^2}{gk + k^2 V_0^2}$$

$$v^2 = \frac{g^2 + gkV_0^2 - g^2}{k(gk + k^2 V_0^2)}$$

$$= \frac{gV_0^2}{gk + k^2 V_0^2} = \frac{\frac{g}{k} V_0^2}{g/k + V_0^2}$$

$$V_t = \sqrt{\frac{g}{k}} \qquad v^2 = \frac{V_t^2 V_0^2}{V_t^2 + V_0^2} \qquad v = \sqrt{\frac{V_t^2 V_0^2}{V_t^2 + V_0^2}}$$