HW # 4 Solutions

Q-1 A simple harmonic oscillator consists of a 150 g mass attached to a spring whose force constant is $10^4 \frac{dy}{cm}$. The mass is displaced by 2.5 cm and released from rest. Calculate

(a) the natural frequency ν_0 and the period τ_0.

(b) the total energy and the maximum speed.

SHO $\ddot{x} + \omega_0^2 x = 0$

$\omega_0^2 = k/m \qquad k = 10^4\ dy/cm \qquad m = 150\ g$

$1\ dy = 10^{-5}\ N$

$= 10^{-5}\ kg\, m\, s^{-2}$

maximum amplitude $= 2.5\ cm$

The solution $x = A \cos(\omega_0 t + \phi)$

$\dot{x} = A\omega \sin(\omega t + \phi)$

at $t=0$ $\dot{x} = 0$ $\dot{x}_0 = A\omega_0 \sin \omega_0 t$

(a) natural frequency $= \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4\ dy/cm}{150 g}} = \sqrt{\frac{10^4 \times 10^{-5}\ kg\, m\, s^{-2}}{150 \times 10^{-3}\ kg\ 10^{-2}\ m}}$

$= \sqrt{\frac{10^{-1}}{150} \times 10^3 \times 10^2\ s^{-2}}$

The period $\tau_0 = \frac{2\pi}{\omega_0}$

$= \frac{2\pi}{8.1849}\ s = 0.76\ s$

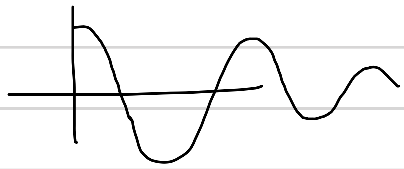
(b) The total energy of the SHO $= \frac{1}{2} k A^2 = \frac{1}{2}\, 10^4 \times 10^{-5} \frac{N}{10^{-2} m} \times (2.5 \times 10^{-2})^2\ m^2$

$= \frac{1}{2}\, 10^{-3} \times 2.5^2\ N m = 3.125 \times 10^{-3}\ J$

Maximum Speed $= A\omega_0 = 2.5\ cm \times 8.1849\ s^{-1} = 20\ cm s^{-1}$

Q-2 Allow the motion in the preceding problem in a resisting medium. After oscillating for 8 s, the maximum amplitude decreases to half of the initial value. Calculate
(a) the damping parameter β
(b) the frequency ν_1 and compare it with the undamped frequency
(c) and the decrement of the motion.

The position co-ordinate of the damped harmonic oscillator can be written as $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$



Now we are considering 2 maximum points.

$A_1 = A_0 \qquad A_0 = Ae^{-\beta t}\cos(\omega_1 t - \delta) \qquad = Ae^{-\beta t}$

$A_2 = A_0/2 \qquad A_0/2 = Ae^{-\beta(t+T_1)}\cos(\omega_1(t+T_1) - \delta) = Ae^{-\beta(t+T_1)}$

$$\frac{A_0}{A_0/2} = \frac{1}{e^{-\beta T_1}}$$

$$2 = e^{\beta T_1}$$

$$\ln 2 = \beta T_1 \qquad T_1 = 8s$$

$$\beta = \frac{1}{8s}\ln 2 = 0.0866\ s^{-1} = 8.66 \times 10^{-2}\ s^{-1}$$

(b) Now we know the damping parameter β and also we know the natural frequency ω_0 of the system.

Angular frequency of the damping SHO $\omega_1^2 = \omega_0^2 - \beta^2$ ——— (1)

$$= (8.1649^2 - 0.0866^2)$$

$$\omega_1 = 8.1644$$

It is very lightly damped

$$\tau_1 = 2\pi/\omega_1 = 0.7695\ s$$

(c) Decrement is defined as

$$Exp[\beta\tau_1] = Exp[0.0866 * 0.7695] = 1.06891$$

$\beta = 0.0866\ s^{-1}$

$\tau_1 = 0.7695\ s$

Q-3 The oscillator in the problem 1 is set in to the motion by giving it an initial velocity 2 cm/s at its equilibrium position. Calculate

(a) the maximum displacement, and

(b) the maximum potential energy.

$$x(t) = A\sin(\omega_0 t - \phi)$$
$$\dot{x}(t) = A\omega_0\cos(\omega_0 t - \phi)$$

So the maximum velocity = $A\omega_0$

$$2\,\frac{cm}{s} = A\omega_0 \quad \leftarrow A \text{ is the maximum amplitude.}$$

$\omega_0 = 8.16497$

$$A = \frac{2\,cm\,s^{-1}}{8.18497} = 0.244\,cm$$

Notice that we reached the answer in 2 methods.

One can also obtain the same solution by considering energy. If at equilibrium the SHO is given a v_0 speed, at the maximum displacement, the total of this kinetic energy is transferred to the potential energy.

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx_0^2$$

$$x_0 = \frac{\sqrt{m}}{\sqrt{k}}v_0 = v_0/\omega_0$$

(b) the maximum potential energy can be calculated by $P.E_{max} = \frac{1}{2}kx_0^2$

where $x_0 \longrightarrow$ the maximum displacement.

$$PE = \frac{1}{2}\left(1\times10^4\,\frac{dyne}{cm}\right)(0.244\,cm)^2$$

$$= \frac{1}{2}\times\frac{10^4\times10^{-5}}{10^{-2}}\times(0.244\times10^{-2})\,\frac{Nm^2}{m}$$

$$= 0.00297\,J = 29.7\times10^{-4}\,J$$

Q4 Consider a simple harmonic oscillator. Calculate the time averages of the kinetic and potential energies over one cycle, and show that these quantities are equal. Why is this a reasonable result. Next calculate the space averages of kinetic and potential energies. Discuss the results.

(a) Time Averages.

$$x = A\cos\omega_0 t \qquad T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_0^2\sin^2\omega_0 t$$

$$\dot{x} = A\omega_0\sin\omega_0 t \qquad U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2\omega_0 t$$

$$\langle T\rangle_{time} = \frac{1}{\tau}\int_0^\tau \frac{1}{2}mA^2\omega_0^2\sin^2\omega_0 t\,dt = \frac{1}{2\tau}mA^2\omega_0^2\int_0^\tau \sin^2\omega_0 t\,dt = \frac{1}{4}mA^2\omega_0^2$$

$$\langle U\rangle_{time} = \frac{1}{\tau}\int_0^\tau \frac{1}{2}kA^2\cos^2\omega_0 t = \frac{1}{2}kA^2\int_0^\tau \cos^2\omega_0 t\,dt \quad \text{—— (1)}$$

$$\langle T\rangle_{time} = \frac{1}{2}mA^2\omega_0^2\int_0^\tau \sin^2\omega_0 t\,dt = \frac{1}{2}\frac{mA^2}{\tau}\frac{k}{m}\int_0^\tau \sin^2\omega_0 t\,dt$$

$$\langle T\rangle_{time} = \frac{1}{2}\frac{kA^2}{\tau}\int_0^\tau \sin^2\omega_0 t\,dt \quad \text{—— (2)}$$

By (1) & (2) $\langle T\rangle_{time} = \langle U\rangle_{time} = \frac{1}{4}mA^2\omega_0^2$

It is clear that $\langle T\rangle_{time} = \langle U\rangle_{time}$. It is clear that position and the velocity change with the time in the similar fashion. So the K.E and Potential energies also change in the similar way, thus the time averages would be the same.

(b) Position Average

$$\langle U\rangle_x = \frac{1}{2A}k\int_0^A x^2\,dx = \frac{1}{2A}k\left[\frac{x^3}{3}\right]_0^A = \frac{kA^2}{6}$$

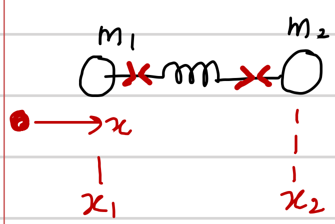
$$\langle T\rangle_t = \frac{1}{A}\int_0^A \frac{1}{2}m\dot{x}^2\,dx \qquad \dot{x} = A\omega_0\sin\omega_0 t \qquad \dot{x}^2 = A^2\omega_0^2\sin^2\omega_0 t = A^2\omega_0^2[1-\cos^2\omega_0 t]$$

$$\dot{x}^2 = A^2\omega_0^2\left[1-\frac{x^2}{A^2}\right]$$

$$= \frac{m}{2A}\int_0^A A^2\omega_0^2\left(1-\frac{x^2}{A^2}\right)dx = \frac{m}{2A}\left[A^2\omega_0^2\right]\left[x-\frac{x^3}{3A^2}\right]_0^A = \frac{mA\omega_0^2}{2}\left[A-\frac{A}{3}\right]$$

$$= \frac{mA^2\omega_0^2}{2}\frac{2}{3} = \frac{1}{3}mA^2\omega_0^2$$

Q-5 Two masses $m_1 = 90\,g$ and $m_2 = 160\,g$ slides freely in a horizontal frictionless track by a spring whose force constant $k = 0.5\,N/m$. Find the frequency of oscilliatory motion for this system. Can you explain this motion.



$m_1 = 90\,g$
$m_2 = 160\,g$

Let's pick the coordinate x_1 & x_2 for the two masses with relative to a fixed origin.

New length of the spring $x_2 - x_1$

Natural length $\rightarrow l$

extension of the spring $\rightarrow x_2 - x_1 - l$

Now we can write the equation of motion, for both masses.

$$m_1 \ddot{x}_1 = k(x_2 - x_1 - l) \quad \text{—— (1)}$$
$$m_2 \ddot{x}_2 = -k(x_2 - x_1 - l) \quad \text{—— (2)}$$

(2) $\rightarrow$ $k x_1 = m_2 \ddot{x}_2 + k x_2 - kl$

$$x_1 = \frac{1}{k}[m_2 \ddot{x}_2 + k x_2 - kl]$$

$$\frac{m_1}{k}\frac{d^2}{dt^2}[m_2 \ddot{x}_2 + k x_2 - kl] = k\, x_2 - [m_2 \ddot{x}_2 + k x_2 - kl] - kl$$

$$\frac{m_1}{k}\frac{d^2}{dt^2}\left[m_2 \ddot{x}_2 + k x_2\right] = -m_2 \ddot{x}_2$$

$$\frac{d^2}{dt^2}\left[m_1 m_2 \ddot{x}_2 + k m_1 x_2\right] = -k m_2 \ddot{x}_2$$

$$\frac{d^2}{dt^2}\left[m_1 m_2 \ddot{x}_2 + k(m_1 + m_2) x_2\right] = 0$$

x_2 is already a function of time. If $\ddot{x}_2$ is non zero, the only possibility that this equation can be zero is

$$m_1 m_2 \ddot{x}_2 + k(m_1 + m_2) x_2 = 0$$
$$\ddot{x}_2 + \frac{k(m_1 + m_2)}{m_1 m_2} x_2 = 0$$

That means x_1 oscillates with a frequency ω where

$\omega^2 = \dfrac{k(m_1 + m_2)}{m_1 m_2}$ where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is defined as the reduced mass of the system.

Q-b In class we proved that the solution of the critically damped oscillator to be $x(t) = (A+Bt)e^{-\beta t}$. Show that $x(t) = Bt\,e^{-\beta t}$ satisfies the equation of motion for the critically damped oscillator.

Let's start with the equation of motion for a critically damped system:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad \text{—— (1)}$$

$$x(t) = Bte^{-\beta t}$$
$$\dot{x}(t) = Be^{-\beta t} - B\beta te^{-\beta t} = Be^{-\beta t}[1-\beta t]$$
$$\ddot{x}(t) = -B\beta e^{-\beta t} - B\beta e^{-\beta t} - B\beta t(-\beta)e^{-\beta t}$$
$$= B\beta^2 te^{-\beta t} - 2B\beta e^{-\beta t}$$

Substituting in (1)

$$B\beta^2 te^{-\beta t} - 2B\beta e^{-\beta t} + 2\beta Be^{-\beta t} - 2B\beta^2 te^{-\beta t} + \omega_0^2 Bte^{-\beta t} = 0$$

$$B\beta^2 t - 2B\beta + 2B\beta - 2B\beta^2 t + \omega_0^2 Bt = 0$$

$$-B\beta^2 t + \omega_0^2 Bt = 0$$
$$Bt[\omega_0^2 - \beta^2] = 0$$

for the critically damped condition, $\omega_0^2 = \beta^2$, So $x(t) = Bte^{-\beta t}$ satisfy the equation of motion for the damped harmonic oscillator.