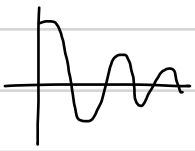
HW # 5 Solutions

Q-1 The amplitude of an oscillator changes from 8 mm to 3 mm in 200 s. What is the value of the damping constant β of the system.?



For the damped oscillator, $x(t) = Ae^{-\beta t}\cos(\omega t+\phi)$

at $t=t_1$ $x=x_1 = 8$ mm

at $t=t_1+t_0$ $x=x_2=3$ mm

$$x_1 = Ae^{-\beta t_1}\cos(\omega t_1+\phi) \quad \text{——} \; ①$$

$$x_2 = Ae^{-\beta(t_1+t_0)}\cos(\omega t_1+\omega t_0+\phi)$$

$\omega t_0 = 2n\pi$ (Because we are looking at two maximas here)

$$x_2 = Ae^{-\beta(t_1+t_0)}\cos(\omega t_1+\phi) \quad \text{——} \; ②$$

$①/② \Rightarrow \quad \dfrac{x_1}{x_2} = \dfrac{1}{e^{-\beta t_0}}$

$$\frac{8}{3} = e^{\beta t_0}$$

$$\beta t_0 = \ln 8/3$$

$$\beta = \frac{1}{200\,s}\ln\frac{8}{3} = 0.004\ s^{-1}$$

Q-2 Consider a harmonic oscillator driven by a periodic force given by $F(t) = 3\cos\omega t + 5\cos 3\omega t$.

In this equation, $\omega = \omega_0/2$ where ω_0 is the natural frequency of the system. The damping factor of the system $\beta = 0.2\omega_0$. Find the equation of the steady state of the system.

In class we solve the problem $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$ and found the steady state solution as

$$x(t) = D(\omega)\cos(\omega t - \delta) \quad \text{——} \; ①$$

where $D(\omega) = \dfrac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta^2}}$ & $\tan\delta = \left(\dfrac{2\omega\beta}{\omega_0^2-\omega^2}\right)$

In the current problem, we have $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 3\cos\omega t + 5\cos 3\omega t$

What we are going to do here is consider the linear super position $\longrightarrow$

and re-write the problem as

$$\ddot{x}_1 + 2\beta \dot{x}_1 + \omega_0^2 x_1 = 3 \cos \omega t$$
$$\ddot{x}_2 + 2\beta \dot{x}_2 + \omega_0^2 x_2 = 5 \cos 3\omega t$$

& solve for $x_1(t)$ & $x_2(t)$ at the steady state. Then the total solution can be written as,

$$x(t) = x_1(t) + x_2(t)$$

$\omega = \omega_0/2$

$$x_1(t) = \frac{3}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta^2}} \cos(\omega t - \delta_1)$$

$$= \frac{3}{\sqrt{\left(\omega_0^2 - \frac{\omega_0^2}{4}\right)^2 + 4\left(\frac{\omega_0}{2}\right)^2 (0.2\omega_0)^2}} \cos\left(\frac{\omega_0}{2}t - \delta_1\right)$$

$$= \frac{3}{\sqrt{\frac{9\omega_0^4}{16} + 0.04\,\omega_0^4}} \cos\left(\frac{\omega_0}{2} - \delta_1\right) = \frac{3}{\sqrt{\frac{9}{16}+0.04}\;\omega_0^2} \cos\left(\frac{\omega_0 t}{2} - \delta_1\right)$$

$$x_2(t) = \frac{5}{[(\omega_0^2 - 9\omega^2)^2 + 4(3\omega)^2(0.2\omega_0)^2]} \cos\left(\frac{\omega_0}{2}t - \delta_2\right)$$

$\boxed{Q\text{-}3}$ The damping factor of a spring suspension system is $0.25\,\omega_0$. Find the resonant frequency and the phase angle for this system, when it is driven by an external driving force with a frequency $\omega = \omega_0/3$.

Resonant Frequency $\omega_R = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{\omega_0^2 - (0.25)^2\omega_0^2}$
$= \sqrt{1-(0.25)^2}\;\omega_0 = 0.9682\,\omega_0$

Now if the system is driven by a force with a frequency $\omega = \omega_0/3$

$$\tan\delta = \frac{2\omega\beta}{\omega_0^2 - \omega^2} = \frac{2\,\frac{\omega_0}{3}\cdot\frac{\omega_0}{4}}{\omega_0^2 - \omega_0^2/9} = \frac{\omega_0^2/6}{8\omega_0^2/9} = \frac{\omega_0^2}{\cancel{2}6} \times \frac{\cancel{9}3}{8\omega_0^2}$$

$$= 3/16$$

$$\delta = 11.12^\circ$$

Q-4 Given that the amplitude of the damped harmonic oscilliator drops to $1/e$ of its initial value after n complete cycles. Show that the ratio of period of oscillation to the period of oscillation with no damping is given by $\left(1+\frac{1}{4\pi^2 n^2}\right)^{-1}$. } Please note there is a typo in HW.

$$A_1 = A e^{-\beta t} \cos(\omega_0 t - \delta)$$

$$A_2 = A e^{-\beta(t+nT_d)} \cos(\omega_0(t+nT_d) - \delta)$$

$$= A e^{-\beta(t+nT_D)} \cos(\omega_0 t - \delta)$$

$$\frac{A_1}{A_2} = e^{\beta n T_D} = \frac{1}{1/e} = e$$

$$\beta n T_d = 1$$

$$T_d = \frac{2\pi}{\omega_1} \qquad \beta n \frac{2\pi}{\omega_1} = 1$$

$$\frac{\omega_1}{\beta} = \frac{1}{2\pi n}$$

$$\omega_1 = 2\pi\beta n$$

$$\omega_1^2 = \omega_0^2 - \beta^2$$

$$\left(\frac{\omega_1}{\omega_0}\right)^2 = 1 - \frac{\beta^2}{\omega_1^2}$$

$$\left(\frac{\omega_1}{\omega_0}\right)^2 = 1 - \left(\frac{1}{2\pi n}\right)^2$$

$$\left(\frac{\omega_1}{\omega_0}\right)^2 = 1 - \frac{1}{4\pi^2 n^2}$$

$$\frac{\omega_1}{\omega_0} = \sqrt{1 - \frac{1}{4\pi^2 n^2}}$$

$$\frac{T_d}{T_1} = \left(1 - \frac{1}{4\pi^2 n^2}\right)^{-\frac{1}{2}}$$