Classical Mechanics - PHYS 310 - Fall 2013 Lec 12: Supplementary Notes

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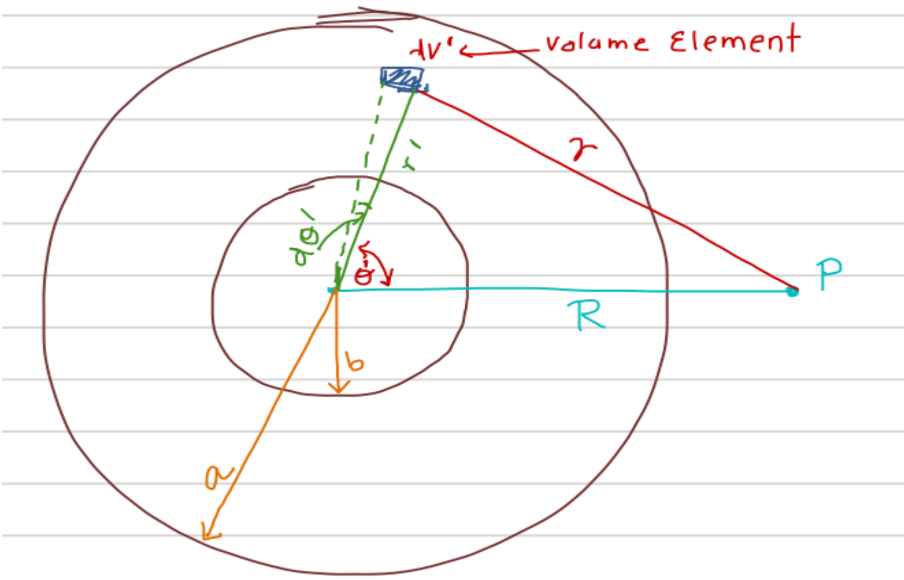
Gravitational Potential due to a Spherical Shell



Let's look at the above figure We want to find the potential at a point P at a distance R from the center of the sphere, due the the spherical Shell.

Now we know how to find the potential due to a point mass, so we consider a point like mass element with a volume dV' at a distance r from the point P:

Let's say this mass element is at a distance r' from the center O. We can write the potential due to the mass element as:

$$d\phi = -\frac{GM}{r} \tag{1}$$

where,

$$M = \rho(r')dv'$$

So

$$d\phi - \frac{G\rho(r')dv'}{r} \tag{2}$$

Notice that, we used the primed coordinate system for the integration variable.

Now,we want to write the volume element dv' in spherical coordinates.

$$dv' = r'^2 dr' Sin\theta d\theta d\phi \tag{3}$$

Now

$$d\phi = -G\int \frac{\rho(r')r'^2 dr' Sin\theta}{r} d\theta d\phi \tag{4}$$

The integrations should cover the whole mass distribution.

We can easily do the integration over ϕ, as we do not see any ϕ dependance in the integrand.

$$d\phi = -G2\pi \int \frac{\int \rho(r')r'^2 dr' Sin\theta d\theta}{r} \tag{5}$$

Let's assume a uniform distribution of mass, that means the mass density $\rho(r')$ is a constant.

$$d\phi = -G2\pi\rho\frac{\int r'^2 dr' Sin\theta d\theta}{r} \tag{6}$$

We need to be careful, when we are doing the θ and r integrations. Because r, the distance between the element mass and the point P we are interested in depend on both θ and r'.

By considering the geometry of the picture, we can write an expression for r:

$$r^2 = r'^2 + R^2 - 2r'RCos\theta \tag{7}$$

For a given r', by differentiating the above equation, we get:

$$2rdr = 2r'RSin\theta d\theta \tag{8}$$

$$\frac{Sin\theta}{r}d\theta = \frac{dr}{r'R} \tag{9}$$

Now, by substituting this in the above equation:

$$d\phi = -2\pi G\int r'^2 dr'\frac{dr}{r'R} \tag{10}$$

which then reduces to:

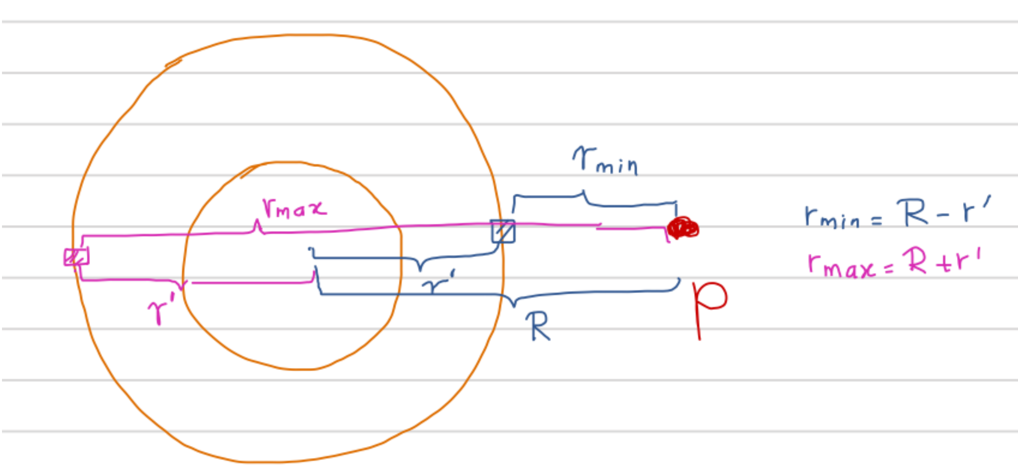
$$d\phi = -\frac{2\pi\rho G}{R}\int r'drdr' \tag{11}$$

We can see that the r' integration runs to takes care of the whole mass. That means, r' goes from b to a. On the other hand r integration is dependent.

In fact, the limits of r depends on where the point P is:

$$d\phi = -\frac{2\pi\rho G}{R}\int_b^2 r'dr'\int_{r_{min}}^{r_{max}} dr \tag{12}$$

Case I: The point P is outside the sphere



$r_{min} = R - r'$ and $r_{max} = R + r'$

$$
\begin{aligned}
\phi(r > a) &= -\frac{2\pi\rho G}{R}\int_b^a r'dr'\int_{R-r'}^{R+r'} dr \\
&= -\frac{2\pi\rho G}{R}\int_b^a r'dr'\left[R + r' - (R - r')\right] \\
&= -\frac{2\pi\rho G}{R}\int_b^a r'dr'2r' \\
&= -\frac{4\pi\rho G}{R}\int_b^a r'^2 dr' \\
&= -\frac{4\pi\rho G}{R}\left[\frac{r'^3}{3}\right]_b^a \\
&= -\frac{4\pi\rho G}{R}\frac{\left[a^3 - b^3\right]}{3}
\end{aligned}
$$

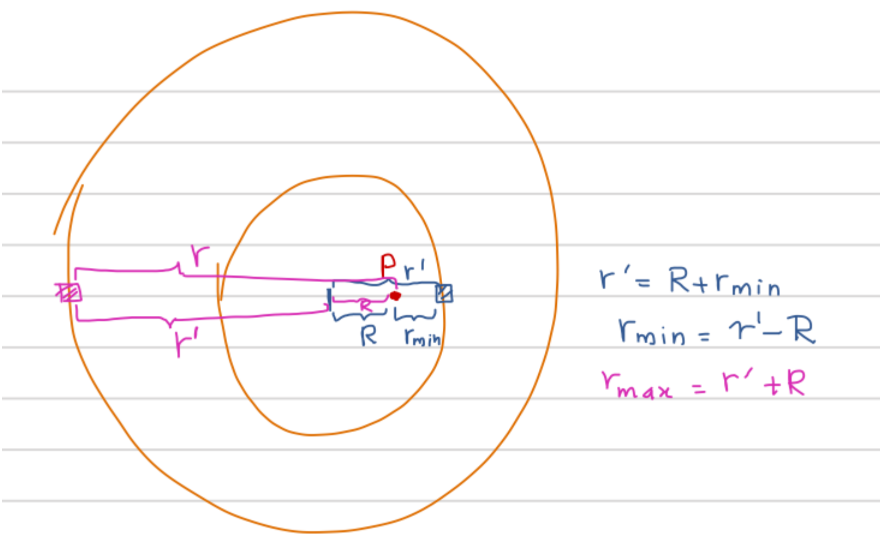
Note that $M = \frac{4}{3}\pi\rho\left[a^3 - b^3\right]$ and

$$
\phi = -\frac{GM}{R} \tag{13}
$$

When the point P is outside the shell, we can still consider the whole mass as a point mass.

Case II: The point P is within the core: $R < b$

Loo at the figure. $r_{min} = r' - R$ and $r_{max} = r' + R$:

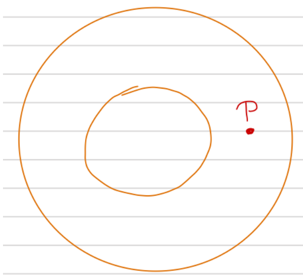


$$
\begin{aligned}
\phi(R < b) &= -\frac{2\pi\rho G}{R}\int_b^a r'dr'\int_{r'-R}^{r'+R} dr \\
&= -\frac{2\pi\rho G}{R}\int_b^a r'dr'2R && (14) \\
&= -4\pi\rho G\int_b^a r'dr' && (15) \\
&= -2\pi\rho G\left[a^2 - b^2\right] && (16)
\end{aligned}
$$

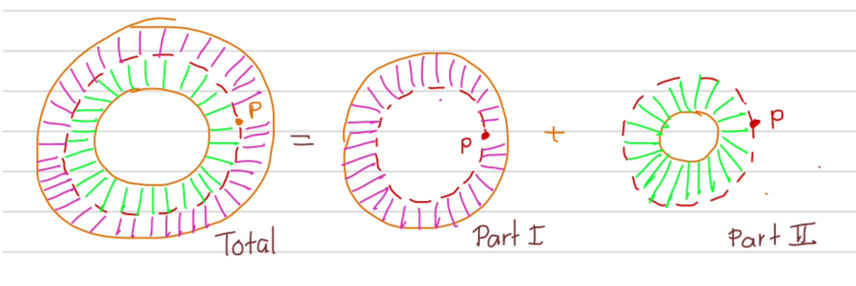
There is no dependance on R, where the point P is. In fact, when the point is inside, the potential within the core is a constant.

Case III: The point P is within the shell : $b < R < a$

When the point P is within the shell, (which is shown in the figure)



We can think of this problem basically in two parts as shown in the following diagram.



For the part I in the problem above, we can think of the problem equivalent to the case II above, the point P is onside the core. So all we have to do is use the same equations as case II (eq.21), but instead of having the limits of the integration as b to a, we will have b to R.

which gives,

$$\phi_{partI} = -2\pi G\rho \left(a^2 - R^2\right) \tag{17}$$

In the similar way, part II can be obtained from the result of Case I above, with the change in integration limits.

which gives:

$$-\frac{4\pi\rho G}{3R}\left(R^3 - b^3\right) \tag{18}$$

Altogether, this give the gravitational potential at a point P (where b<R<a),

$$\phi = -\frac{4\pi\rho G}{3R}\left(R^3 - b^3\right) - 2\pi\rho G\left(a^2 - R^2\right) \tag{19}$$

By simplifying this equation, we get:

$$\phi_{b<R<a} = -4\pi\rho G\left(\frac{a^2}{2} - \frac{b^3}{3R} - \frac{R^2}{6}\right) \tag{20}$$

We basically got the three following equations for the gravitational potential due to a spherical shell:

$$\phi_{R<b} = -2\pi\rho G\left[a^2 - b^2\right] \tag{21}$$

$$\phi_{b<R<a} = -4\pi\rho G\left(\frac{a^2}{2} - \frac{b^3}{3R} - \frac{R^2}{6}\right) \tag{22}$$

$$\phi = -\frac{GM}{R} \tag{23}$$

When you look at the equation (21, 22, and 23), three are few important things to note. For example, when you make $R \to a$ in the equation 22, we get the same as eq23. Simlarly, if you make $R \to b$ in the eq. (22), we get thte eq. 21.

In fact, what we are seeing here is the continuity of the gravitational potential. That is physical. If we ha a discontinuity in the gravitational potential at the interfaces, then, it is not possible to define the force at the interface, which is the gradient of the gravitational potential.

Anyway, the bottom line, is now due to the spherical shell of mass, we know the gravitational potential all around space.

Few things to note:

- when the point P is outside the distribution, all the mass can be considered as a point Mass concentrated at the center of the spherical shell
- We we look at the force within the core, the potential is constant everywhere, that means, no force is acting on the mass, which sits with in the core, and no work is needed to move the mass from one point to another within the core.

Interesting thing to note: What is Dark Matter

Within this model we discussed above, we can calculate different data related to celestial bodies. However, the result from the above model and the experimental observation do not agree completely, There is a qualitative agreement. But now quantitative agreement.

Because of this, Astrophysicists conclude that for many galaxies, there must be matter other than that observed, and this "unobserved matter" often called as "Dark Matter", is a research area in Astro Physics.