Classical Mechanics - PHYS 310 - Fall 2013 Lec 12: Supplementary Notes

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CALCULATING THE GRAVITATIONAL POTENTIAL

In the las lecture, we have discussed the gravitational potential due to spherical shell of mass. We have use the basic definition of the gravitational potential due to a mass element as $\phi = -G\int \frac{dm}{r}$, which is then simplified to $\phi = -G\int_V \frac{\rho(r')}{r}dv'$

In this lecture, we take an alternative approach to the same problem. In order to calculate to gravitational force on a mass m due to another mass, we can consider the gravitational field line. In fact, the gravitational potential due to a object is related to the density of gravitational field lines.

We can calculate the gravitational flux going through an arbitrary surface S.

$$\phi_m = \int_S \hat{n}.\hat{g}da \tag{1}$$

Notice that the vector $\hat{n}$ is normal to the surface.$\hat{g}$ points where ever the gravitational field is pointing to. So $\hat{n}.\hat{g}$, we can get the component of the gravitational field vector perpendicular to the surface. Let's see what this surface integration equals to.

If we think about the gravitational field intensity due to a mass M,

$$\hat{g} = -\frac{GM}{r^2}\hat{r} \tag{2}$$

With that, we can now calculate the gravitational flus across the surface.

$$\begin{aligned}
\phi_m &= \int_S \hat{n}.\hat{g}da &(3)\\
&= -\int_S \hat{r}.\frac{GM}{r^2}\hat{r}da &(4)\\
&= -\int \frac{GM}{r^2}r^2 Sin\,\theta\ d\theta\ d\phi &(5)\\
&= -4\pi GM &(6)
\end{aligned}$$

Now by combining the eq. (1) and eq. (6), we get:

$$\hat{n}.\hat{g}da = -4\pi GM \tag{7}$$

According to the eq.(7), the mass enclosed by the surface is what matters, It does not matter how the mass is distributed or where the mass is located.

If we have a desecrate mass distribution, we have;

$$\hat{n}.\hat{g}da = -4\pi G\sum_i m_i \tag{8}$$

If we have a continuous mass distribution:

$$\hat{n}.\hat{g}da = -4\pi G\int_V \rho(r)dv \tag{9}$$

We can use the Poisson's equation in any of the above forms. We can however convert it in to other forms:

Notice that, when we calculate the gravitational flux go through a surface, we have done an integration over the whole surface. We basically look at the component of gravitational field vector perpendicular to the surface through a surface elements, and then sum it over the whole surface.

Instead, one can look through the whole volume covered by the surface, and look at how diverging the gravitational field vector is. In particular, the surface integration of the component of gravitational field vector can be converted in to an integration of the divergence of the gravitational field vector across the whole volume.

Diverging field. Picture Here xxxxxxxxxxxxxxxxxx

Mathematically, this can be written as:

$$\int_S \hat{n}.\hat{g} da = \int_V \vec{\nabla}.\hat{g} dV \tag{10}$$

Equation (10) is called the divergence theorem. Vy combining the equations (9) and (10), we can write,

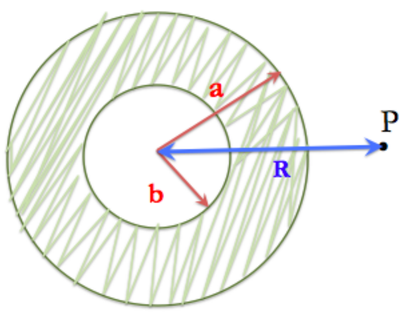
$$\int_S \hat{n}.\hat{g} da = -\int_V 4\pi G\rho(r) dV \tag{11}$$

which gives:

$$\vec{\nabla}.\hat{g} = -4\pi G\rho \tag{12}$$

Let's do an example.

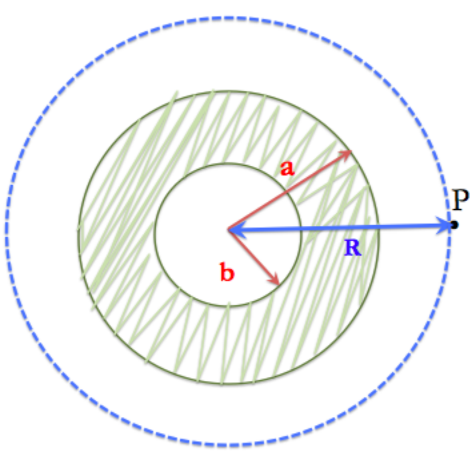
Calculating the Gravitational field vector due a Spherical Shell type Mass Distribution



I need to calculate the gravitational potential due to a spherical wheel type mass distribution as shown in the figure. The total mass of the shell is M and the density is ρ.

Case I R >a

Let's consider a closed surface with radius R as shown in Blue in the following figure.



Let's calculate the gravitational field vector at a point P as shown in the figure. Let's say the gravitational field vector in the region is g_{out}. Notice that according to the choice our gaussian surface, the gravitational field vector in perpendicular to the surface. In face $\hat{n}$ and $\hat{g}_{out}$ are point ting in the same direction. So $\hat{n}.\hat{g}_{out} = g_{out}$

We can now use the Poisson's equation:

$$\int_S \hat{n}.\hat{g} da = -4\pi G M_{enclosed} \tag{13}$$

which reduces to:

$$\int_S g_{out} da = -4\pi G M \tag{14}$$

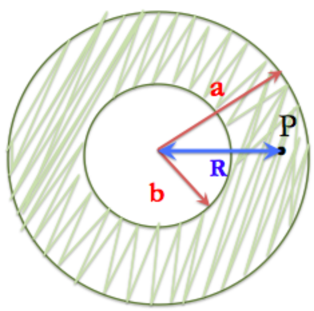
$$g_{out} 4\pi R^2 = -4\pi G M \tag{15}$$

$$g_{out} = -\frac{GM}{R^2} \tag{16}$$

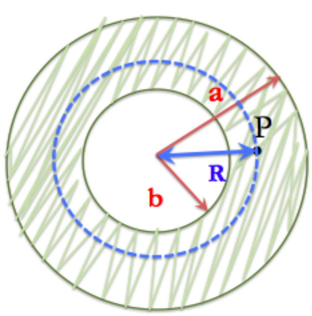
Equation (16) is similar to that was obtained using the gravitational potential method (equation 5.22 in the text book).

Case I b<R <a

Now let's look at the point P in side the shell area as shown in the following figure.



Let's consider a closed surface with a spherical symmetry and radius R as shown (in blue) in the figure.



Now we can use the Poisson's equation:

$$\int_S \hat{n}.\hat{g} da = -4\pi G M_{enclosed} \tag{17}$$

$$g_{mid} 4\pi R^2 = -4\pi G \left[\frac{4}{3}\left(R^3 - b^3\right)\right]\rho \tag{18}$$

$$g_{mid} = -\frac{4\pi G\rho}{3R^2}\left(R^3 - b^3\right) \tag{19}$$

$$g_{mid} = \frac{4\pi G\rho}{3}\left(\frac{b^3}{R^2} - R\right) \tag{20}$$

Equation (20) is similar to that was obtained using the gravitational potential method (equation 5.22 in the text book).

Case I R>b

Now let's look at the point P in side the shell area as shown in the following figure.

Let's consider a closed surface with a spherical symmetry and radius R as shown (in blue) in the figure.

Now we can use the Poisson's equation:

$$\int_S \hat{n}.\hat{g}da = -4\pi G M_{enclosed} \tag{21}$$

There 's no mass enclosed by the spherical surface shown in blue. So:

$$\int_S \hat{n}.\hat{g}da = 0 \tag{22}$$

which gives, $\hat{g}_{in} = 0$

In summary, we have calculate the gravitational field vector due to a spherical shell of mass,M, and we got:

$$g_{out} = -\frac{GM}{R^2} \tag{23}$$

$$g_{mid} = \frac{4\pi G\rho}{3}\left(\frac{b^3}{R^2} - R\right) \tag{24}$$

$$g_{in} = 0 \tag{25}$$

The above equation is exactly equals to the equation (5.22) from the text, which was obtained by evaluating the gravitational potential first. Now let's see, starting with the equation (23),(24), and (25), if we can evaluate the gravitational potential.

Let's evaluate the gravitational potential, starting with the gravitational field vector.

EVALUATING THE GRAVITATIONAL POTENTIAL FROM $\hat{g}$

Case I r>a

Let's start with the definiton:

$$\hat{g} = \vec{\nabla}\phi \tag{26}$$

For the case, where $\hat{g}$ is only a function of r, we can write this as:

$$\hat{g} = -\frac{d}{dr}\phi \tag{27}$$

by integrating:

$$-\int_{\infty}^{r} d\phi = \int_{\infty}^{r} \hat{g}.dr \tag{28}$$

$$-\phi(r > a) = \int_{\infty}^{r} -\frac{GM}{r^2}dr \tag{29}$$

$$-\phi(r > a) = \left[\frac{GM}{r}\right]_{\infty}^{r} \tag{30}$$

Which gives:
$$\phi(r > a) = -\frac{GM}{r} \tag{31}$$
which is equivalent to the equation (5.21) from the text book.
Case II b<r<a
$$\hat{g} = -\frac{d}{dr}\phi \tag{32}$$
$$-\int_{\infty}^{r} d\phi = \int_{\infty}^{r} \hat{g}.dr \tag{33}$$
$$-\int_{\infty}^{r} d\phi = \int_{\infty}^{a} \hat{g}_{out}.dr + \int_{a}^{r} \hat{g}_{mid}.dr \tag{34}$$
$$-\phi = \frac{GM}{r}|_{\infty}^{a} + \int_{a}^{r} \left(\frac{4\pi G\rho}{3} \left(\frac{b^3}{r^2} - r \right) \right) dr \tag{35}$$
$$-\phi = \frac{GM}{a} + \left[\frac{4\pi G\rho}{3} \left(-\frac{b^3}{r} - \frac{r^2}{2} \right) \right]_{a}^{r} \tag{36}$$
$$-\phi = \frac{GM}{a} + \frac{4\pi G\rho}{3} \left[\frac{b^3}{a} + \frac{a^2}{2} - \frac{b^3}{r} - \frac{a^2}{2} \right] \tag{37}$$
$$-\phi = \frac{4\pi \left(a^3 - b^3\right) \rho G}{3a} + \frac{4\pi G\rho}{3} \left[\frac{b^3}{a} + \frac{a^2}{2} - \frac{b^3}{r} - \frac{r^2}{2} \right] \tag{38}$$
$$-\phi = \frac{4\pi G\rho}{3} \left[\frac{a^3}{a} - \frac{b^3}{a} + \frac{b^3}{a} + \frac{a^2}{2} - \frac{b^3}{r} - \frac{r^2}{2} \right] \tag{39}$$
$$-\phi = \frac{4\pi G\rho}{3} \left[\frac{3a^3}{2} - \frac{b^3}{r} - \frac{r^2}{2} \right] \tag{40}$$
$$\phi = -\frac{4\pi G\rho}{3} \left[\frac{3a^3}{2} - \frac{b^3}{r} - \frac{r^2}{2} \right] \tag{41}$$
This is exactly sillier to the equation we obtained in the equation 5.22 in the text.
In the similar way, we can calculate the gravitational potential with the core region.