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EXAMPLES OF THE CALCULUS OF VARIATION

In the last lecture, we discussed how to find the conditions for extremizing an integral quantity, such as the distance between two points, time interval etc. The solutions for extremizing the quantity

$$J = \int_{x_1}^{x_2} f\left\{y(x), y'(x); x\right\} dx \tag{1}$$

as,

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \tag{2}$$

We applied the Euler's equation to show that the shortest path between two points is a straight line. Today, we proceed with more examples.

EXAMPLE: BRACHISTOCHORNE PROBLEM

Bachistro mean short and chrome means time. That is here we are trying to find the shortest time between two incidents. A particle s moving in a constant force field starting at rest from (x_1, y_1) to some other point (x_2, y_2). FInd the party that allows the particle to come in the least amount of time.

First thing: Let's define what needs to be conserved. The time needs to be extremized.

$$t = \int_{x_1}^{x_2} \frac{ds}{v} \tag{3}$$

ds is the elemental length on the path and v is the instantaneous velocity within the length segment ds.

Now we need to write an equation for the v. For that, we can consider the conservation of energy. We are basically designing a surface, such that the time of travel is the shortest. It is not a free fall. We can use the conservation of energy to find the speed of the particle at a given point.

$$\frac{1}{2}mv^2 = mgx \tag{4}$$

So

$$v = \sqrt{2gx} \tag{5}$$

By substituting this in eq. (3):

$$t = \int_{x_1}^{x_2} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gx}} \tag{6}$$

$$t = \int_{x_1}^{x_2} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2gx}} dx \tag{7}$$

$$t = \int_{x_1}^{x_2} \sqrt{\frac{1 + (y')^2}{2gx}} dx \tag{8}$$

Now we can minimize the time using the Euler's equations. Just comparing the quantity we wrote int he Euler's equation,

$$f = \sqrt{\left(\frac{1 + y'^2}{x}\right)} \tag{9}$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \tag{10}$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \tag{11}$$

$$\frac{\partial f}{\partial y'} = Constant = a \tag{12}$$

$$\frac{1}{\sqrt{x(1+y'^2)}}2y' = a \tag{13}$$

$$\frac{y'^2}{x((1+y'^2)} = b \tag{14}$$

$$y'^2(1-bx) = bx \tag{15}$$

$$y'^2 = \frac{bx}{1-bx} \tag{16}$$

$$y'^2 = \frac{x}{c-x} \tag{17}$$

$$dy = \sqrt{\frac{x}{c-x}}dx \tag{18}$$

In order to solve this equation, let's take a solution of the form:

$$x = \frac{c}{2}(1-Cos\theta) = c(Sin^2\theta/2) \tag{19}$$

$$dx = cSin\theta/2Cos\theta/2d\theta \tag{20}$$

$$dy = \sqrt{\frac{cSin^2\theta/2}{c - c\,Sin^2\theta/2}}c\,Sin\theta/2Cos\theta/2d\theta \tag{21}$$

$$dy = cSin^2\theta/2d\theta \tag{22}$$

$$\int dy = \frac{c}{2}\int (1-Cos\theta)\,d\theta$$

$$y = \frac{c}{2}\left[\theta - Sin\theta\right]$$

EXAMPLE: SURFACE GENERATED BY REVOLVING A LINE CONNECTING TWO FIXED POINTS

Consider the surface generated by revolving a line connecting two fixed points (x_1, y_1), about an axis coplanar with the two points. Find the equation of the line connecting the points such that the surface area generated by the revolution. (i.e. the area of the surface of revolution) is a minimum.

Now let's assume that the line connecting (x_1, y_1) and (x_2, y_2) passing around the y axis, and coplanar with the two points.

Let's consider an elemental area, which is formed by rotating an ds area around the yaxis.

$$dA = 2\pi x ds = 2\pi x\sqrt{dx^2 + dy^2} \tag{23}$$

$$dA = 2\pi x\sqrt{dx^2 + dy^2} \tag{24}$$

$$dA = 2\pi x\sqrt{1+y'^2}dx \tag{25}$$

Now by integrating this quantity, we can achieve the total area, and then we are trying to find the mimim area.

$$A = \int_{x_1}^{x_2} 2\pi x \sqrt{1 + y'^2} dx \tag{26}$$

In order to find the extremum value of A, we use the Euler's equation:

$$f = 2\pi x \sqrt{1 + y'^2} \tag{27}$$

Then we use,

$$\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \tag{28}$$

Here $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial y'} = 2\pi x \frac{1}{2} \frac{2y'}{\sqrt{1+y'^2}}$

Then we get,

$$\frac{xy'}{\sqrt{1 + y'^2}} = Constant = C \tag{29}$$

$$y' = \frac{c}{\sqrt{x^2 - c^2}} \tag{30}$$

$$\frac{dy}{dx} = \int_{x_1}^{x_2} \frac{c}{\sqrt{x^2 - a^2}} \tag{31}$$

By integrating, we can easily solve for $y(x)$

THE SAME PROBLEM IN A DIFFERENT PROSPECTIVE

Let's do the same problem in a different point of view. This will allow you to understand the choice of independent coordinate

In the previous problem, choose two points $x_1, y_1)$ and (x_2, y_2) joined by a curve y(x). We need to find $y(x)$, such that if we revolve the curve around the x-axis, the surface area is a minimum.

Now with the same geometry analysis, we can take

$$dA = 2\pi y ds == 2\pi y \sqrt{dx^2 + dy^2} = 2\pi y \sqrt{1 + y'^2})dx \tag{32}$$

$$f = 2\pi y \sqrt{1 + y'^2}$$
$$\frac{\partial f}{\partial y} = 2\pi \sqrt{1 + y'^2}$$
$$\frac{\partial f}{\partial y'} = 2\pi y \frac{1}{2} \frac{2y'}{\sqrt{1 + y'^2}}$$

Now by plugging these in the Euler's equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$
$$2\pi \sqrt{1 + y'^2} - \frac{d}{dx} 2\pi y \frac{y'}{\sqrt{1 + y'^2}} = 0$$
$$\sqrt{1 + y'^2} = \frac{d}{dx} \frac{yy'}{\sqrt{1 + y'^2}}$$

Why did this get complicated" In the previous case, it was very easy to solve. Did we do a mistake. Can we fix it?

Let's talk about that in the next class: