Classical Mechanics - PHYS 310 - Fall 2013 Lec 19: Supplementary Notes
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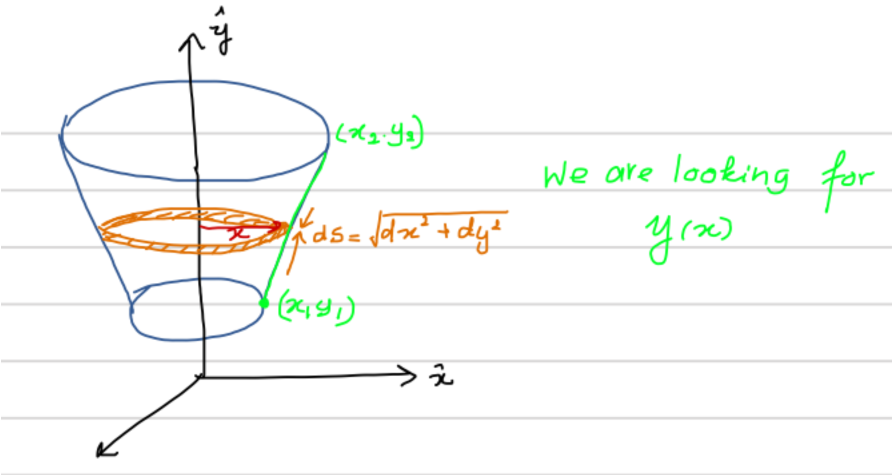
EXAMPLES OF THE CALCULUS OF VARIATION

In the last lecture, we tried to evaluate the shape of a wire, which gives the minimum surface area by evolving it around an axis. We approached it in two different ways. First, we assume that the wire arc revolves around the y axis and second we assumed that the wire revolves around the x axis.

It is the similar problem. For the first problem, we could easily reach an answer, But for the second problem, we could not reach an answer easily.

Method I::

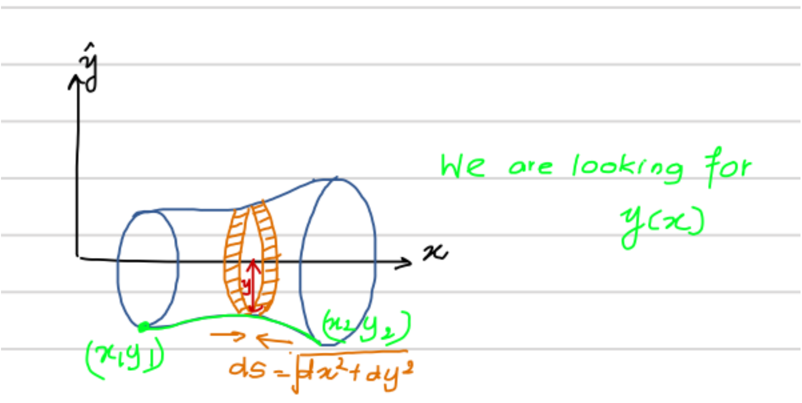


The areas is written as $A = \int_{x_1}^{x_2} 2\pi x ds$ and by minimizing the surface area, we get,

$$\frac{dy}{dx} = \int_{x_1}^{x_2} \frac{c}{\sqrt{x^2 - c^2}} \quad (1)$$

It is then easy to find what is the y as a function of x.

Method II:



The areas is written as $A = \int_{x_1}^{x_2} 2\pi y ds$ and by minimizing the surface area, we get,

$$\sqrt{1+y'^2} = \frac{d}{dx}\frac{yy'}{\sqrt{1+y'^2}} \tag{2}$$

It is clear that, the equation (1) is easier to solve than the equation (2).

What Happens in the above two problems:

In the method I, we have the functional f which is not explicitly depend on x, so $\frac{\partial f}{\partial y} = 0$

Which result in: $\frac{d}{dx}\frac{\partial f}{\partial y'} = 0$ thus, $\frac{\partial f}{\partial y'} = 0$.

But in the method II,

$$\frac{\partial f}{\partial y} \neq 0$$

So, instead, we could have used y as the independent quantity, and look for x(y), instead of y(x).

However, for all the problems, it is not easier to figure out what is the best choice for independent variable before hand. It is some times a trial and error procedure.

If we have chosen x as the independent coordinate, and if $\frac{\partial f}{\partial y} \neq 0$

but

$\frac{\partial f}{\partial x} = 0$

Is there a better way of using the Euler's Equations?

THE SECOND FORM OF EULER'S EQUATION

The second form of Euler's Equations is convenient if the functional f does not explicitty depend on x,

ie. $\frac{\partial f}{\partial x} = 0$

Let's consider,

$$\begin{aligned} \frac{d}{dx} f\left\{y, y'; x\right\} &= \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x} \\ &= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x} \end{aligned} \tag{3}$$

I want to get rid of y'':

Consider:

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \tag{4}$$

By combining eq (3) and eq.(4)

$$\begin{aligned} \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) &= \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial x} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \\ &= \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) \end{aligned} \tag{5}$$

In the eq.(5), the term in the parenthesis is zero according to the Euler's equation. Which then simplifies to:

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x}$$

We can write this as:

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0 \tag{6}$$

Now according to the equation (6), if $\frac{\partial f}{\partial x} = 0$,we can find,

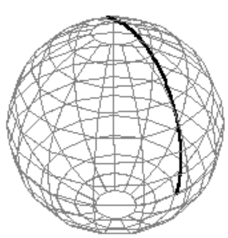
$$f - y'\frac{\partial f}{\partial y'} = Constant \tag{7}$$

Now let's look at the previous problem with the Second form of Euler's Equation:

$$\begin{aligned} f = y\sqrt{1+y'^2} \quad \rightarrow \frac{\partial f}{\partial x} &= 0 && (8)\\ f - y'\frac{\partial f}{\partial y'} &= c \\ y\sqrt{1+y'^2} - y'\frac{2yy'}{2\sqrt{1+y'^2}} &= c && (9)\\ \frac{y + yy'^2 - yy'^2}{\sqrt{1+y'^2}} &= c \\ \frac{y}{\sqrt{1+y'^2}} &= c \\ \frac{y^2}{1+y'^2} &= d \end{aligned}$$

EXAMPLE

A geodesic is a line that represents the shortest path between any two points when the path is restricted to a particular surface. FInd the geodesic on a sphere.



In general, we write the arc length in 3D space (in spherical coordinates as

$$ds = \sqrt{dr^2 + r^2 d\theta^2 + r^2 Sin^2\theta d\phi^2} \tag{10}$$

Now when it says, we are looking for a geodesic on a sphere of radius $r = \rho$, $dr = 0$.
With that constrain condition, we have the arc length on the sphere as:

$$ds = \sqrt{\rho^2 d\theta^2 + \rho^2 Sin^2\theta d\phi^2} \tag{11}$$

$$ds = \rho\sqrt{d\theta^2 + Sin^2\theta d\phi^2} \tag{12}$$

Now we can find the arc length by integrating this quantity.

$$
\begin{aligned}
s &= \int ds = \int \rho\sqrt{d\theta^2 + Sin^2\theta d\phi^2} \\
s &= \int \rho\sqrt{\left(\frac{d\theta}{d\phi}\right)^2 + Sin^2\theta}\ d\phi \\
&= \int \rho\sqrt{\left(\theta'\right)^2 + Sin^2\theta}\ d\phi
\end{aligned}
$$

Now we are looking for minimizing s. By comparing with the previous proof, we set

$$f = \sqrt{\left(\theta'\right)^2 + Sin^2\theta} \tag{13}$$

We have chosen the independent coordinate as ϕ, Noe we notice that $\frac{\partial f}{\partial phi} = 0$. So it is easier if we use the Euler's equation of the second form.

$$f - \theta'\frac{\partial f}{\partial \theta'} = a \tag{14}$$

$$
\begin{aligned}
\sqrt{\left(\theta'\right)^2 + Sin^2\theta} - \theta'\frac{\partial}{\partial \theta'}\sqrt{\left(\theta'\right)^2 + Sin^2\theta} &= a \qquad (15)\\
\sqrt{\left(\theta'\right)^2 + Sin^2\theta} - \theta'\frac{1}{2}\frac{1}{\sqrt{\left(\theta'\right)^2 + Sin^2\theta}}2\theta' &= a \\
Sin^2\theta &= a\sqrt{\left(\theta'\right)^2 + Sin^2\theta} \\
Sin^4\theta &= a^2\left(\left(\theta'\right)^2 + Sin^2\theta\right) \\
Sin^2\theta(Sin^2\theta - a^2) &= a^2\theta'^2 \\
\left(\frac{d\theta}{d\phi}\right)^2 &= \frac{Sin^2\theta(Sin^2\theta - a^2)}{a^2} \\
\left(\frac{d\phi}{d\theta}\right)^2 &= \frac{a^2}{Sin^2\theta(Sin^2\theta - a^2)} \\
\left(\frac{d\phi}{d\theta}\right) &= \frac{aCosec^2\theta}{\sqrt{1 - a^2Cosec^2\theta}}
\end{aligned}
$$

EXAMPLE: THIS PROBLEM WILL APPEAR IN HOMEWORK # 7

(a) FInd the curve $y(x)$ that passes through the points $(0,0)$ and $(1,1)$ and minimizes the functional $I[y] = \int_0^1 \left(\left(\frac{dy}{dx}\right)^2 - y^2\right) dx$.

(b) What is the minimum value of the integral?

(c) Evaluate $I[y]$ for a straight line $y = x$ between the two points $(0,0$ and $(1,1)$.