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FUNCTIONS WITH MORE THAN ONE DEPENDENT VARIABLES

In the last lectures, we discussed the Euler's equation. In all the cases we discussed, there's one independent variable and one independent variable. We found the type of a dependent function $y(x)$ such that the integral $\int f\{y, y'; x\}\, dx$ is minimized. Now we want to extend this to the case of more than one dependent function.

For example, if we are looking at the dynamics of a two particle system, and we are looking at the dynamics of both the particles as a function of time, t.

The independent variable is t, the dependent variables are $x1(t), y1(t), z1(t)$ and $x2(t), y2(t), z2(t)$. That means compared to the previous functional $J = \int_{x_1}^{x_2} f\{y, y'; x\}\, dx$, we have

$$J = \int_{x_1}^{x_2} f\left\{y_1(x), y_1^{'}(x), y_2(x), y_2^{'}(x).....; x\right\} dx \tag{1}$$

where

$$f\left\{y_1(x), y_1^{'}(x), y_2(x), y_2^{'}(x).....; x\right\} \tag{2}$$

Let's use the short-hand notation:

$$f \to f\left\{y_i(x), y_i^{'}(x); x\right\} for\ i = 1, 2, ..., n \tag{3}$$

Now $y_1, y_2, ..$ do not depend on each other.

Let's recall how did we deal with the situation when we had only one dependent function.

$$y(\alpha, x) = y(0, x) + \alpha\eta(x) \tag{4}$$

Then we could write the quantity $\frac{dJ}{d\alpha}$,

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x)dx \tag{5}$$

We can follow the same arguments for more than one dependent variables:

By defining a variational parameter, (Here notice that we are basically looking at individual variations)

$$y_(\alpha, x) = y_i(0, x) + \alpha\eta_i(x) \tag{6}$$

and get the equation,

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i}\right)\eta_i(x)dx \tag{7}$$

So the above equation is true if we have :

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0\ \ for\ all\ i \tag{8}$$

example Find out the condition to minimize the integral $\int_{x_1}^{x_2} \left(y_1 y_2^2 + y_1^2 y_2 + y_1' y_2'\right) dx$

$$\frac{d}{dx}\frac{\partial f}{\partial y_1'} - \frac{\partial f}{\partial y_1} = 0 \tag{9}$$

$$\frac{d}{dx}\frac{\partial f}{\partial y_2'} - \frac{\partial f}{\partial y_2} = 0 \tag{10}$$

EULER'S EQUATION WHEN AUXILIARY CONDITIONS ARE IMPOSED

Well....... We have done this problem..... Remember, in the case we calculated the geodesic on a sphere. All what we did there was used$r = R$. and use that equation to simplify the integral equation. In fact, we use the auxiliary condition to reduce the number of coordinates from 3 to 2. We were able to do that, because the auxiliary condition is very simple.

Let's imagine that constraint equation is not that simple. Some thing like $r^2 - a\sqrt{z^2 + r^2} = 0$ ..... (Please do not ask me what kind of physical constraint that is, I just made up some thing)

How do we impose such a condition and minimize an integral. Another way of asking the same question is, How do we use the Euler's equation, when such complicated Auxiliary conditions are imposed.

Let's consider the case, where we have only two dependent variable $y(x)$, and $z(x)$. But they are dependent by an Auxiliary condition.

$$f\left\{y_i, y_i^{'}; x\right\} \to f\left\{y, y', z, z'; x\right\} \tag{11}$$

When we had one dependent function $y(x)$, we had:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial \alpha} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]\frac{\partial y}{\partial \alpha} dx = 0 \tag{12}$$

and we then immediately said:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \tag{13}$$

which is the Euler's equation.

Now for the functional (eq. 11), we can use the variational parameter and write :

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\frac{\partial y}{\partial \alpha}\right] dx = 0 \tag{14}$$

Now if $y(x)$ and $z(x)$ are not related, we can simply use the Euler's equations. But from eq.(14), we cannot deduce the the Euler's equation.

Let's think about the additional equation we had

In addition to the equation(14), w which needs to be satisfied in order to minimize J, we have the auxiliary condition.

Let's write the auxiliary condition in general as:

$$g\left\{y_i; x\right\} = 0 \tag{15}$$

In this particular case with $y(x)$ and $z(x)$ as the dependent variables, we have,

$$g\left\{y, y', z, z'; x\right\} = 0 \tag{16}$$

This condition relates $\frac{\partial y}{\partial \alpha}$ and $\frac{\partial z}{\partial \alpha}$ in the above equation (14) related. That means the two parts in the paranthesis of the above eq.(14) does not separately go to zero.

Now, let's try to see the relationship between s $\frac{\partial y}{\partial \alpha}$ and $\frac{\partial z}{\partial \alpha}$.

We start with the auxiliary condition"

$$g\left\{y, z; x\right\} = 0 \tag{17}$$

$$dg = \left[\frac{\partial g}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial \alpha}\right] d\alpha = 0 \tag{18}$$

Now we will introducevariational functions:

$$\begin{aligned} y(\alpha, x) &= y(0, x) + \alpha\eta_1(x) \qquad (19) \\ y(\alpha, x) &= y(0, x) + \alpha\eta_2(x) \end{aligned}$$

By putting these variational functions in the above eq.(18),

$$\left[\frac{\partial g}{\partial y}\alpha\eta_1(x) + \frac{\partial g}{\partial z}\alpha\eta_2(x)\right] = 0 \tag{20}$$

$$\frac{\eta_1(x)}{\eta_2(x)} = -\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}} \tag{21}$$

We can re-write the eq.(14) as:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\eta_2(x)\right]\eta_1(x)dx = 0 \tag{22}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\frac{\eta_2(x)}{\eta_2(x)}\right]\eta_1(x)dx = 0 \tag{23}$$

Combinning the eq.(21) and eq.(23), we get

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(-\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}}\right)\right]\eta_1(x)dx = 0 \tag{24}$$

In order to satisfy the equation (24), the quality in the square parenthesis should go to zero.

$$\left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(-\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}}\right)\right] = 0 \tag{25}$$

which can then be written as:

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1} \tag{26}$$

The left hand side of the is equation involves only derivatives of f and g with respect to y and y', and the right hand side involves derivatives with respect to z and z'. Now because both y and z are functions of$x$, the two side of the equation may only be a function of x

Let's then call:

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1} = -\lambda(x) \tag{27}$$

which then will be reduced to:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} = 0 \tag{28}$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} = 0 \tag{29}$$

Now, we need to find three quantities $y(x)$, $z(x)$ and $\lambda(x)$.

We need to have three equations. which are given by the above two equations and the constraint equation.

We will do some examples in the next class.