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Of all possible paths along which a dynamical system may move from one point to another in configuration space is whin a specified time interval, the actual path that which minimizes the time integral of the Lagrangean function for the system.

Using the generalized coordinates, we can write the Euler-Lagrangean equation.

$$\frac{\partial f}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \tag{1}$$

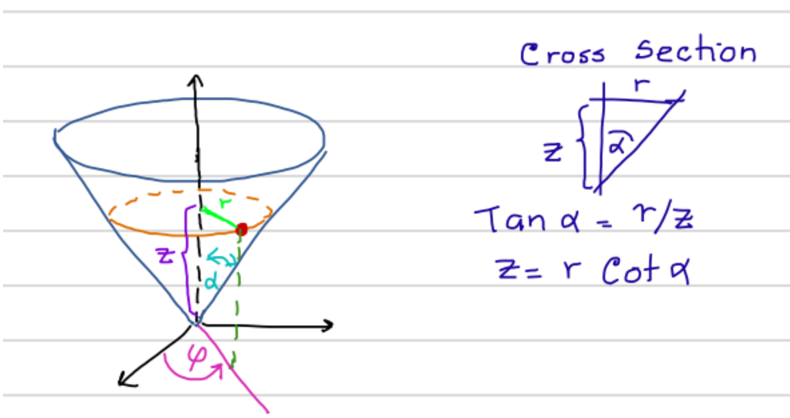
for $j = 1, 2, ...s$ where s is the degrees of freedom in the system

EXAMPLE

A particle of mass m is constrained to move on the inside surface of a smooth cone of half angle α. The particle is subject to a gravitational force. Determine a set of generalized coordinates and determine the constraints. Find the Lagrange's equation of motion.

OK, the first task is to find out a good set of coordinate system which is suitable for explaining the dynamics of the system. The particle is moving on the surface of a cone. The cylindrical coordinate system is convenient for explaining this motion.

So we pick $(r\theta, z)$ in explaining the problem. Then the question is are they independent? Let's look at the problem.



As it is shown in the cross set ion of the above figure, if you are at a particular z value, then your r coordinate is defined. Because the particle has to stay on the surface of the cone. In fact, the constraint condition can be mathematically written as,

$$tan\alpha = \frac{r}{z} \ , \ \rightarrow z = rcot\alpha \tag{2}$$

Now we can pick the set of generalized coordinates as:
(z, α) or (r, α)
Here, we are going to pick it as (r, α).
Now let's write the Lagrangean in terms of the Generalized coordinates $L(q_j, \dot{q}_j)$.
In cylindrical coordinate system, the v^2 can be written as:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 \tag{3}$$

That is interns of (r, θ, z) coordinates. Now, we convert this in to generalized coordinates (r, θ). In fact, we are doing to get substitute $z = rcot\alpha$. and $\dot{z} = \dot{r}cot\alpha$

$$\begin{aligned} v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}cot^2\alpha \\ v^2 &= \dot{r}^2\left(1 + cot^2\alpha\right) + r^2\dot{\theta}^2 \\ &= \dot{r}^2Cosec^2\alpha + r^2\dot{\theta}^2 \end{aligned} \tag{4}$$

and the potential energy

$$U = mgz = mgrcot\alpha \tag{5}$$

So the Lagrangean can be written as:

$$L = \frac{1}{2}m\dot{r}^2Cosec^2\alpha + r^2\dot{\theta}^2 - mgrcot\alpha \tag{6}$$

Now we can apply the Euler-Lagrangean Equations,with respect to all the generalized coordinates.
w.r.t. θ:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\ \& \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} &= 0 \end{aligned}$$

Notice that, we did not write an equation w.r.t. z: Let's then simplify Euler-Lagrangean Equations.

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\ \frac{\partial L}{\partial \theta} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \\ \rightarrow mr^2\dot{\theta} = Constant \end{aligned} \tag{7}$$

If you think about this last result, $mr^2\dot{\theta} = constant$ tells some thing is conserved. Do you recognize this conserved quantity. It is the angular momentum of the particle about the z axis.

Now let's simplify the Euler-Lagrnagean equation with respect to r:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \\ \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mgcot\alpha \\ \frac{\partial L}{\partial \dot{r}} = m\dot{r}cosec^2\alpha \end{aligned} \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{9}$$
$$\ddot{r}Cosec^2\alpha - \left[mr\dot{\theta}^2 - mgcot\alpha\right] = 0$$
$$\ddot{r}cosec^2\alpha - mr\dot{\theta}^2 + mgcot\alpha = 0$$
$$\ddot{r} - r\dot{\theta}^2 sin^2\alpha + g\ sin\,\alpha\ cos\,\alpha = 0$$

Now we have the complete solution for the particle moving on the surface of a cone:

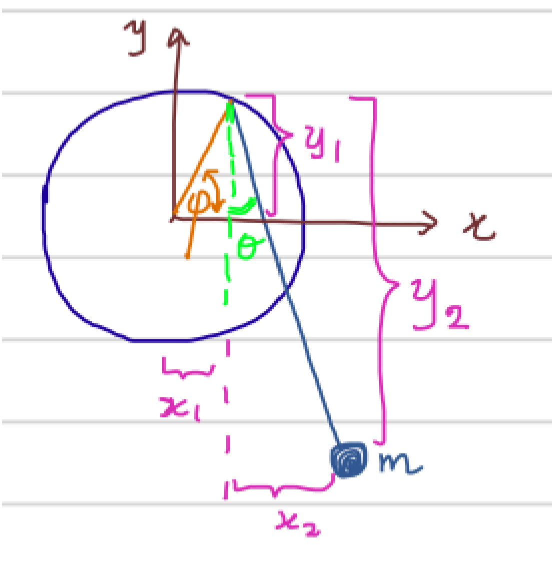
$$mr^2\dot{\theta} = constant$$
$$\ddot{r} - r\dot{\theta}^2 sin^2\alpha + g\ sin\,\alpha\ cos\,\alpha = 0$$

These are two coupled equations. We can solve to find how r and θ evolve with time . That is what are $r(t)$ and $\theta(t)$.

EXAMPLE

The point of support of a simple pendulum of length b moves on a massless rim of radius a rotating with a constant angular velocity ω. Obtain the expression for the Cartesian components of the velocity and accelleration of mass m. Obtain also the angular acceleration for the angle θ shown in the figure.

Let's first make a sketch for the problem.



We need to set up a coordinate system for the mass m. The coordinate system for mass m can be easily written with respect to the point P, which is moving. What we are going to do is, consider the origin as shown in the figure, then write the coordinates of the point P with relative to the origin O, and then write sown the coordinates of the mass m with relative to the moving point P.

Now what are the good coordinate system for describing the motion of point P w.r.t O and the motion of mass m w.r.t. the point P.

The point P is always fixed on the rim, that is with a constant radius a. That mass m is attached to a string of length l, in both cases, we see, if we use the polar coordinates, we can use the constraint that the radial coordinate is fixed. (If we had used the cartesian coordinate system, the equations of constraints would be complicated)

Now the problem says the rim is moving with a constant angular velocity. That means, $\frac{d\phi}{dt} = \omega \quad \rightarrow \phi = \omega t + c$ and by making the initial conditions such that $c = 0$ we have $\phi = \omega t$

So finally, we get that only generalized coordinate to explain this problem is θ as shown in the figure. Now, we are going to write the transformation equations.

$$\begin{aligned} x &= aCos\omega t + bSin\theta \\ y &= aSin\omega t - bCos\theta \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{x} &= -a\omega Sin\omega t_b\dot{\theta}Cos\theta \\ \dot{y} &= a\omega Cos\omega t + b\dot{\theta}Sin\theta \end{aligned} \tag{11}$$

We can then write the Kinetic Energy of mass m as,

$$\begin{aligned} K.E. &= \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2\right] \\ &= \frac{1}{2}m\left[\left(-a\omega Sin\omega t + b\dot{\theta}Cos\theta\right)^2 + \left(a\omega Cos\omega t + b\dot{\theta}Sin\theta\right)^2\right] \\ &= \frac{1}{2}m\left[a^2\omega^2Sin^2\omega t - 2a\omega b\dot{\theta}Sin\omega tCos\theta + b^2\dot{\theta}^2Cos^2\theta + a^2\omega^2Cos^2\omega t + 2a\omega b\dot{\theta}Sin\theta Cos\omega t + b^2\dot{\theta}^2Sin^2\theta\right] \\ &= \frac{1}{2}m\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2a\omega b\dot{\theta}\left(Sin\theta Cos\omega t - Cos\theta Sin\omega t\right)\right] \\ K.E. &= \frac{1}{2}m\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2a\omega b\dot{\theta}Sin(\theta - \omega t)\right] \end{aligned} \tag{12}$$

and we have the Kinetic Energy as,

$$V = mgy = mg\left[aSin\omega t - bCos\theta\right] \tag{13}$$

And we have the Lagrangean as,

$$L = \frac{1}{2}m\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2a\omega b\dot{\theta}Sin(\theta - \omega t)\right] - mg\left[aSin\omega t - bCos\theta\right] \tag{14}$$

Now let's set up the Euler Lagrangean Equations.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} &= 0 \\ \frac{\partial L}{\partial\theta} &= \frac{1}{2}m\left[2a\omega b\dot{\theta}Cos(\theta - \omega t)\right] + mgb(Sin\theta) \\ \frac{\partial L}{\partial\theta} &= ma\omega b\dot{\theta}Cos(\theta - \omega t) - mgbSin\theta \\ \frac{\partial L}{\partial\dot{\theta}} &= \frac{1}{2}m\left[2b^2\dot{\theta} + 2a\omega bSin(\theta - \omega t)\right] \\ &= mb^2\dot{\theta} + ma\omega bSin(\theta - \omega t) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) &= mb^2\ddot{\theta} + ma\omega b(\dot{\theta} - \omega)Cos(\theta - \omega t) \end{aligned} \tag{15}$$

Now putting every thing together in the Euler-Lagrangean Equation:

$$\begin{aligned} mb^2\ddot{\theta} + ma\omega b(\dot{\theta} - \omega)Cos(\theta - \omega t) - ma\omega b\dot{\theta}Cos(\theta - \omega t) + mgbSin\theta = 0 \\ mb^2\ddot{\theta} - ma\omega^2bCos(\theta - \omega t) + mgbSin\theta = 0 \\ \ddot{\theta} - \frac{a\omega^2}{b}Cos(\theta - \omega t) + \frac{g}{b}Sin\theta = 0 \end{aligned} \tag{16}$$

Now we need to integrate this equation to find θ as a function of time.

When you look at this equation, you can also recognize that, when $\omega = 0$, that means, when the pendulum support is not fixed to the rotating rim, the equation simplifies to that of a simple pendulum, which kind of proves the validity of the above equation.