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**Equivalance of Lagrange's and Newton's Equations**

We have solved a number of problems with both Newtonian Mechanics and Lagrange's Dynamics. In some cases, we solved the same problem with both the methods. And we shoed that both methods idld the same result. In this class, we are going to prove this in general, The result of Newton;s equations is identical to those obtained from the Lagrange's Dynamics.

Let's first prove this for Rectangular coordinates.

## RECTANGULAR COORIDNATES

We can apply the Euler-Lagrangean Equations as:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \tag{1}$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \tag{2}$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0 \tag{3}$$

We can write this collectively as:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{4}$$

for $i = 1, 2, 3$,

Here $x_1 = x$, $x_2 = y$, and $x_3 = z$.

We are going to show that the Lagrange's equations yield the same results as the Newton's Equations.

Let's start with the Lagrange's Equations.

$$\frac{\partial}{\partial x_i}(T - U) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}(T - U) = 0 \tag{5}$$

For this particular case: $T = T(\dot{x}_i)$ and $U = U(x_i)$

This yields:

$$\frac{\partial}{\partial x_i}(-U) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}(T) = 0 \tag{6}$$

$$\frac{\partial U}{\partial x_i} = -\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} \tag{7}$$

Since we ware trying to relate the Lagrangean Dynamics to Newton's Equations, let's relate the potential energy and the force.

For a conservative field:

$$F_i = -\frac{\partial U}{\partial x_i} \tag{8}$$

By combining the above equations, we get,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = F_i \tag{9}$$

Now let's express the kinetic energy in terms of the coordinates:

$$T = \frac{1}{2}m\sum_i \dot{x}_i^2 \tag{10}$$

$$\frac{\partial T}{\partial x_i} = m\dot{x}_i \tag{11}$$

By putting it in the eq.(9),we get,

$$\frac{d}{dt}m\dot{x}_i = F_i \tag{12}$$

which gives,

$$m\ddot{x}_i = F_i \tag{13}$$

Here,we proved that the Lagrange's equation and Newton's Equation yield the same result. But we did it for the specific case of rectangular equations. Now we are going to prove it for the general case. The system motion is explained by a set of generalized coordinates.

## NEWTON'S EQUATIONS AND THE LAGRANGE'S EQUATION USING GENERALIZED COORDINATES

From the previous section, we know that we need to differentiate the Kinetic energy and the potential energy with respect to the generalized coordinate. Let's start writing the Kinetic Energy interns of the generalized coordinates.

$$x_i \rightarrow x_i(q_j, t) \tag{14}$$

$$T = \frac{1}{2}m\sum_i \dot{x}_i^2 \tag{15}$$

Let's differentiate T with respect to $q_j$,

$$\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\frac{1}{2}m\sum_i \dot{x}_i^2 \tag{16}$$

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_i m\dot{x}_i\frac{\partial \dot{x}_i}{\partial \dot{q}_j} \tag{17}$$

Now let's take the time derivative of this:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_i \frac{d}{dt}\left(m\dot{x}_i\frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right) = \sum_i \frac{d}{dt}\left(m\dot{x}_i\frac{\partial x_i}{\partial q_j}\right) \tag{18}$$

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In writing the last equation, we used

$$\begin{aligned} x_i &\rightarrow x_i(q_j, t) \\ \frac{d}{dt}x_i &= \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t} \qquad (19) \\ \frac{\partial \dot{x}_i}{\partial \dot{q}_j} &= \frac{\partial x_i}{\partial q_j} \qquad (20) \end{aligned}$$

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$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \sum_i m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_j} \tag{21}$$

Let's consider the last term of the above equation:

$$\begin{aligned}
m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_j} &= m\dot{x}_i \frac{d}{dt}\left(\frac{\partial}{\partial q_j} x_i(q_k, t)\right) \\
&= \sum_k m\dot{x}_i \frac{\partial}{\partial q_k}\left(\frac{\partial}{\partial q_j} x_i\right)\frac{dq_k}{dt} + m\dot{x}_i \frac{\partial}{\partial t}\frac{\partial x_i}{\partial q_j} \\
&= m\dot{x}_i \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}\right) \\
&= m\dot{x}_i \frac{\partial}{\partial q_j}\left(\dot{x}_i\right) \\
&= \frac{\partial}{\partial q_j}\frac{1}{2} m x_i^2 \qquad (22)
\end{aligned}$$

Putting eq(21) and eq.(22) together,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) &= \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \sum_i \frac{\partial}{\partial q_j}\frac{1}{2} m x_i^2 \qquad (23) \\
&= \sum_i F_i \frac{\partial x_i}{\partial q_j} + \frac{\partial}{\partial q_j} T \qquad (24)
\end{aligned}$$

Now let's consider the work:

$$\begin{aligned}
\delta W &= \sum_i F_i \delta x_i = \sum_i F_i \sum_j \frac{\partial x_i}{\partial q_j}\delta q_j \\
&= \sum_{i,j} F_i \frac{\partial x_i}{\partial q_j}\delta q_j \\
&= \sum_j \left(\sum_i F_i \frac{\partial x_i}{\partial q_j}\right)\delta q_j \\
&= \sum_i Q_j \delta q_j \qquad (25)
\end{aligned}$$

where $Q_j$ is the generalized force:

By combining eq.(24) and eq.(25):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j} T = Q_j \tag{26}$$

We also can relate the generalized force to the potential energy as $Q_j = -\frac{\partial U}{\partial q_j}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j} T = -\frac{\partial U}{\partial q_j} \tag{27}$$

For the conservative fields, the potential energy does not depend on the generalized velocity:

So we can write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}T = -\frac{\partial U}{\partial q_j} \tag{28}$$

which can be written as:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-U) - \frac{\partial}{\partial q_j}(T-U) = 0 \tag{29}$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}L - \frac{\partial}{\partial q_j}L = 0 \tag{30}$$

We proved the Lagrangean Equations utilizing the Newton's equations.
Alright, we dealt with lot of proofs. Let's get back to more problems.

## EXAMPLE

Two blocks of mass $M$ are connected by an extension less, uniform string of length $l$. One block is placed on a smooth horizontal surface, and the other block hangs over the side, the string passes over a frictionless pulley. ascribe the motion of the system (mass of the string is negligible)