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Canonical Equations of Motion and Hamiltonian Dynamics In the last two lectures, we define a new quantity Hamiltonian as :

$$H = \sum_j \dot{q}_j p_j - L \tag{1}$$

For the case where the transformation equations does not have any explicit time dependance, we figured that the above quantity can be written as:

$$H = T + V \tag{2}$$

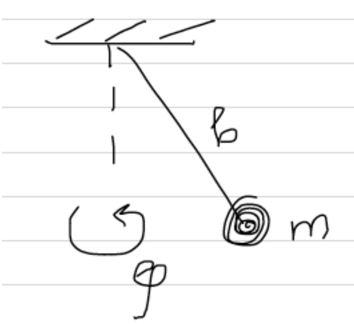
And also we have proved the canonical equations of motion as:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \tag{3}$$

$$-\dot{p}_j = \frac{\partial H}{\partial q_j} \tag{4}$$

EXAMPLE

Use both the Lagrangean method and the Hamiltonian method to find the equations of motion for a spherical pendulum of mass m and length b.



The motion of the spherical pendulum can be explained by (r, θ, ϕ), spherical polar coordinates.
In spherical polar coordinates:

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 Sin^2\theta\dot{\phi}^2 \tag{5}$$

with the constraint $r = b$.
So the Kinetic Energy becomes:

$$T = \frac{1}{2}m\left(b^2\dot{\theta}^2 + b^2 Sin^2\theta\dot{\phi}^2\right) \tag{6}$$

and the Potential Energy

$$V = -mgbCos\theta \tag{7}$$

So the first step is to write the Lagrangean:

$$L = T - V \tag{8}$$
$$L = \frac{1}{2}m\left(b^2\dot{\theta}^2 + b^2Sin^2\theta\dot{\phi}^2\right) + mgbCos\theta \tag{9}$$

Let's first solve this problem using Lagrngean method

Euler-Lagrangean Equation w.r.t. θ

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$
$$nonumber\frac{1}{2}mb^2 2Sin\theta Cos\theta\dot{\phi}^2 - mgbSin\theta - \frac{d}{dt}mb^2\dot{\theta} = 0 \tag{10}$$

$$\ddot{\theta} - Sin\theta Cos\theta\dot{\phi}^2 + \frac{g}{b}Sin\theta = 0 \tag{11}$$

Euler-Lagrangean Equation w.r.t. ϕ

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$
$$\frac{d}{dt}mb^2Sin^2\theta\dot{\phi} = 0$$
$$mb^2Sin^2\theta\dot{\phi} = Constant \tag{12}$$

We can solve for $\theta(t)$ and $\phi(t)$ using the equations (11) and (12)
Now let's solve the same problem using the Canonical equations.

Using Canonical Equations

Since in this problem, the transformation equations do not have any explicit time dependance,

$$H = T + V$$
$$H = \frac{1}{2}m\left(b^2\dot{\theta}^2 + b^2Sin^2\theta\dot{\phi}^2\right) - mgbCos\theta \tag{13}$$

Now we have written the Hamiltonian, and then we need to convert the variables:
Using the eq. (8),

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} \quad \rightarrow \quad \dot{\theta} = \frac{p_\theta}{mb^2} \tag{14}$$
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2Sin^2\theta\dot{\phi} \quad \rightarrow \quad \dot{\phi} = \frac{p_\phi}{mb^2Sin^2\theta} \tag{15}$$

Substituting eq. (14) and (15), in eq. (13), we get:

$$H = \frac{mb^2}{2}\left(\frac{p_\theta}{mb^2}\right)^2 + \frac{mb^2 Sin^2\theta}{2}\left(\frac{p_\phi}{mb^2 Sin^2\theta}\right)^2 - mgbCos\theta \tag{16}$$

$$H = \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2 Sin^2\theta} - mgbSin\theta \tag{17}$$

Now we can apply the Canonical Equations of motion:
Each coordinate has 2 equations:
w.r.t. θ

$$\dot{\theta} = \frac{\partial H}{\partial \theta} = \frac{2p_\theta}{2mb^2} \quad \rightarrow \quad p_\theta = mb^2\dot{\theta} \tag{18}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\left(\frac{p_\phi^2}{2mb^2}\frac{-2}{Sin^3\theta}Cos\theta + mgbSin\theta\right) = \frac{p_\phi^2 Cos\theta}{mb^2 Sin^3\theta} - mgbSin\theta \tag{19}$$

$$\dot{\phi} = \frac{\partial H}{\partial \phi} = \frac{2p_\phi}{2mb^2 Sin^2\theta} \quad \rightarrow \quad p_\phi = mb^2 Sin^2\theta\dot{\phi} \tag{20}$$

$$\dot{p_\phi} = -\frac{\partial H}{\partial \phi} = 0 \quad \rightarrow \quad p_\phi = mb^2 Sin^2\theta\dot{\phi} = Constant \tag{21}$$

By Combining these equations:

$$mb^2\ddot{\theta} = \frac{p_\phi^2 Cos\theta}{mb^2 Sin^3\theta} - mgbSin\theta \tag{22}$$

$$mb^2\ddot{\theta} = \frac{(mb^2 Sin^2\theta\dot{\phi})^2 Cos\theta}{mb^2 Sin^3\theta} - mgbSin\theta \tag{23}$$

$$mb^2\ddot{\theta} = mb^2 Sin\theta Cos\theta\dot{\phi}^2 - mgbSin\theta \tag{24}$$

$$\ddot{\theta} - Sin\theta Cos\theta\dot{\phi}^2 + \frac{g}{b}Sin\theta = 0 \tag{25}$$

We can in fact combine this equation with

$$mb^2 Sin^2\theta\dot{\phi} = Constant \tag{26}$$

$$\dot{\phi} = \frac{k}{Sin^2\theta} \tag{27}$$

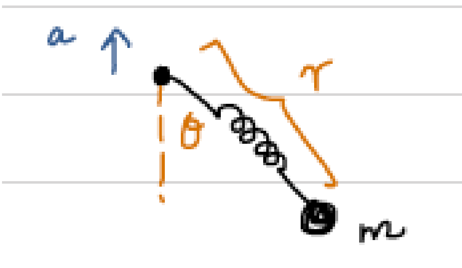
$$\ddot{\theta} - Sin\theta Cos\theta\dot{\phi}^2 + \frac{g}{b}Sin\theta = 0 \tag{28}$$

$$\ddot{\theta} - Sin\theta Cos\theta\frac{k^2}{Sin^4\theta} + \frac{g}{b}Sin\theta = 0 \tag{29}$$

$$\ddot{\theta} - k^2\frac{Cos\theta}{Sin^3\theta} + \frac{g}{b}Sin\theta = 0 \tag{30}$$

EXAMPLE

A pendulum consists of a mass m suspended by a massless spring with untended length b and spring constant k. The pendulum support rises vertically with constant acceleration a.
(a) Use the Lagrangean method to find the equation of motion
(b) Determine the Hamiltonian and Hamilton's equation of motion.
(c) What is the period of small oscillation?



Let's first write the transformation equations.

$$x = rSin\theta \quad \rightarrow \quad \dot{x} = \dot{r}Sin\theta + r\dot{\theta}Cos\theta \tag{31}$$

$$y = \frac{1}{2}at^2 - rCos\theta \quad \rightarrow \quad \dot{y} = at - \dot{r}Cos\theta + r\dot{\theta}Sin\theta \tag{32}$$

We can write the Kinetic energy as:

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{r}Sin\theta + r\dot{\theta}Cos\theta\right)^2 + \left(at - \dot{r}Cos\theta + r\dot{\theta}Sin\theta\right)^2 && (33) \\ &= \frac{1}{2}m\left[\dot{r}^2Sin^2\theta + r^2\dot{\theta}^2Cos^2\theta + 2r\dot{r}\dot{\theta}Sin\theta Cos\theta + a^2t^2 + 2at\left(r\dot{\theta}Sin\theta - \dot{r}Cos\theta\right) + \left(r\dot{\theta}Sin\theta - \dot{r}Cos\theta\right)^2\right] && (34) \\ &= \frac{1}{2}m\left[\dot{r}^2 +^2 \dot{\theta}^2 + a^2t^2 + 2at\left(r\dot{\theta}Sin\theta - \dot{r}Cos\theta\right)\right] && (35) \end{aligned}$$

and the potential energy as:

$$V = mgy + \frac{1}{2}k(r-b)^2 = mg\left(\frac{1}{2}at^2 - rCos\theta\right) + \frac{1}{2}k(r-b)^2 \tag{36}$$

The Lagrangean equation can be written as :

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + a^2t^2 + 2at\left(r\dot{\theta}Sin\theta - \dot{r}Cos\theta\right)\right] - \left[mg\left(\frac{1}{2}at^2 - rCos\theta\right) + \frac{1}{2}k(r-b)^2\right] \tag{37}$$

Now with the Lagrangean, we can write the two Euler-Lagrangean equations: w.r.t θ and w.r.t. r
Now let's set up the Hamiltonian.
Unlike in the previous problems we did, in this case, the transformation equation has an explicit time dependance. So,

$$H \neq T + V \tag{38}$$

So we set up the Hamiltonian as:

$$H = \sum_j p_j\dot{q}_j - L \tag{39}$$

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L \tag{40}$$

Let's find the generalized momenta:

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + a^2t^2 + 2at\left(r\dot{\theta}Sin\theta - \dot{r}Cos\theta\right)\right] - \left[mg\left(\frac{1}{2}at^2 - rCos\theta\right) + \frac{1}{2}k\left(r-b\right)^2\right] \quad (41)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} - matCos\theta \qquad \rightarrow \dot{r} = \frac{p_r}{m} + a\,t\,Cos\theta \quad (42)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + m\,a\,t\,r\,Sin\theta \qquad \rightarrow \dot{\theta} = \frac{p_\theta}{mr^2} - \frac{a\,t}{r}Sin\theta \quad (43)$$

Now we can evaluate the Hamiltonian:

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L \quad (44)$$

$$H = \frac{p_r^2}{m} + p_r atCos\theta - \frac{at}{r}p_\theta Sin\theta + \frac{p_\theta^2}{mr^2} - L \quad (45)$$

By simplifying all the terms:

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{at}{r}p_\theta Sin\theta + a\,tp_r Cos\theta + \frac{1}{2}m\,g\,a\,t^2 - m\,g\,r\,Cos\theta + \frac{1}{2}k(r-b)^2 \quad (46)$$

Now we have west up $H(r, p_r, \theta, p_\theta, t)$, then we can apply the Canonical equations.

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} - \frac{a\,t}{r}Sin\theta \quad (47)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} + a\,t\,Cos\theta \quad (48)$$

Next set of equation:

$$p_r = -\frac{\partial H}{\partial r} \quad (49)$$

$$p_\theta = -\frac{\partial H}{\partial \theta} \quad (50)$$

CONSERVED QUANTITIES

In the last class, along with the canonical equation, we also had another equation:

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad (51)$$

Let's take the total derivative of H:

$$H = \sum_j \dot{q}_j p_j - L \quad (52)$$

$$\frac{dH}{dt} = \sum_j \ddot{q}_j p_j + \dot{q}_j \dot{p}_j - \frac{dL}{dt} \quad (53)$$

$$L \rightarrow L(q_j, \dot{q}_j, t) \quad (54)$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} \quad (55)$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + p_j\ddot{q}_j + \frac{\partial L}{\partial t} \quad (56)$$

Now,

$$\frac{dH}{dt} = \sum_j \ddot{q}_j p_j + \dot{q}_j \dot{p}_j - \left(\sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + p_j \ddot{q}_j + \frac{\partial L}{\partial t} \right) \tag{57}$$

$$\frac{dH}{dt} = \sum_j \dot{q}_j \dot{p}_j - \frac{\partial L}{\partial q_j} \dot{q}_j + \frac{\partial L}{\partial t} \tag{58}$$

$$\frac{dH}{dt} = \sum_j \dot{q}_j \dot{p}_j - \dot{p}_j \dot{q}_j - \frac{\partial L}{\partial t} \tag{59}$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{60}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{61}$$

This tells that, if L does not explicitly depend on time, Hamiltonian is conserved.
If H does not explicitly depend on time, H is conserved.