CENTRAL FORCE MOTION

In the last class, we worked through the equation of motion for a particle in a central force field.

$$F = f(r)\hat{r} \tag{1}$$

Because the Angular momentum is a conserved quantity, we figured that the motion of the particle is constrained to a two-dimensional planar motion.

By applying the Euler-Lagrangean equations, we got;

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(u) \tag{2}$$

where $u = \frac{1}{r}$.

We can also write this in terms of the coordinate r as,

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r) \tag{3}$$

What does this equation say. If you tell me the type of the force function $F(r)$, we can find out $u(\theta)$, which is the type of the orbital. Or else, if you tell me the type of the orbital, I can tell you the type of the force using the above equations. Let's do some examples:

EXAMPLE

A particle in a central force field move in an orbit $r = c\theta^2$. Determine the form of the force function.

$$r = c\theta^2 \tag{4}$$

$$\frac{1}{r} = \frac{1}{c\theta^2} \tag{5}$$

Now we are going to use the equation for the orbit.

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r) \tag{6}$$

Let's do step by step.

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{-1}{r^2}\frac{dr}{d\theta} \tag{7}$$

$$= -\frac{1}{r^2}\frac{d}{d\theta}\left(c\theta^2\right) \tag{8}$$

$$= -\frac{2c\theta}{r^2} = -\frac{2c\theta}{(c\theta^2)^2} \tag{9}$$

$$= -\frac{2}{c\theta^3} \tag{10}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{d}{d\theta}\frac{1}{r}\right) \tag{11}$$

$$= \frac{d}{d\theta}\left(-\frac{2}{c\theta^3}\right) \quad (12)$$

$$= -\frac{2}{c}\frac{-3}{\theta^4} \quad (13)$$

$$= \frac{6}{c\theta^4} \quad (14)$$

Now let's substitute them in the orbital equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r) \quad (15)$$

$$\frac{6}{c\theta^4} + \frac{1}{c\theta^2} = -\frac{\mu r^2 F(r)}{l^2} \quad (16)$$

$$F(r) = -\frac{l^2}{\mu r^2}\left(\frac{6}{c\theta^4} + \frac{1}{c\theta^2}\right) \quad (17)$$

At this point, we have evaluated the force, however, it is written as a function of r and θ.

$$F(r) = -\frac{l^2}{\mu r^2}\left(\frac{6c}{c^2\theta^4} + \frac{1}{c\theta^2}\right) \quad (18)$$

$$= -\frac{l^2}{\mu r^2}\left(\frac{6c}{r^2} + \frac{1}{r}\right) \quad (19)$$

$$= -\frac{l^2}{\mu r^3}\left(\frac{6c}{r} + 1\right) \quad (20)$$

EXAMPLE

Find the force law for a central force field that allows a particle to move in a logarithmic orbit $r = ke^{\alpha\theta}$, where k and α are constants.

$$r = ke^{\alpha\theta} \quad (21)$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\frac{1}{ke^{\alpha\theta}} = -\frac{\alpha}{k}e^{-\alpha\theta} \quad (22)$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(-\frac{\alpha}{k}e^{-\alpha\theta}\right) \quad (23)$$

$$= \frac{\alpha^2}{k}e^{-\alpha\theta} \quad (24)$$

$$= \frac{\alpha^2}{ke^{\alpha\theta}} \quad (25)$$

$$= \frac{\alpha^2}{r} \quad (26)$$

By substituting in the orbital equation:

$$F(r) = -\frac{l^2}{\mu r^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right] \quad (27)$$

$$= -\frac{l^2}{\mu r^2}\left[\frac{\alpha^2}{r} + \frac{1}{r}\right] \quad (28)$$

$$F(r) = -\frac{l^2}{\mu r^2}\left[1 + \alpha^2\right] \quad (29)$$

EXAMPLE

In the previous problem, we know what the orbit looks like $\rightarrow \quad r = ke^{\alpha\theta}$. But, we did not figure out how the r and θ changes with time. That is what we are focusing on now.

$$\dot{\theta} = \frac{l}{\mu r^2} \tag{30}$$

$$\frac{d\theta}{dt} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \tag{31}$$

$$\mu k^2 e^{2\alpha\theta} d\theta = ldt \tag{32}$$

$$\mu k^2 \frac{e^{2\alpha\theta}}{2\alpha} = lt + C1 \tag{33}$$

$$e^{2\alpha\theta} = \frac{2\alpha l}{\mu k^2} t + C1 \tag{34}$$

That gives how θ changes with time:

$$\theta(t) = \frac{1}{2\alpha} ln \left[\frac{2\alpha l}{\mu k^2} t + C1 \right] \tag{35}$$

With that, we can find how the radial coordinate change with time t.

$$r(t) = ke^{\alpha\theta(t)} \tag{36}$$

$$r^2 = k^2 e^{2\alpha\theta} \tag{37}$$

$$= k^2 \left[\frac{2\alpha l}{\mu k^2} t + C1 \right] \tag{38}$$

$$r(t) = \left[\frac{2\alpha l}{\mu} t + k^2 C1 \right]^{\frac{1}{2}} \tag{39}$$

CONSERVATION OF ENERGY IN CENTRAL FORCE FIELD

We are considering non-dissipative systems. So the total energy is constant.

$$E = T + V = Constant \tag{40}$$

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \tag{41}$$

$$= \frac{1}{2}\mu \left(\dot{r}^2 + r^2 \left(\frac{l}{\mu r^2} \right)^2 \right) + U(r) \tag{42}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) \tag{43}$$

We can use this to find out the radial velocity of the problem.

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E - U) - \frac{l^2}{\mu^2 r^2}} \tag{44}$$

This equation tells that $\dot{r}$ vanishes at:

$$\frac{2}{\mu}(E-U)-\frac{l^2}{\mu^2 r^2}=0 \tag{45}$$

That tells, there are two points, (two roots for the above equation for r), where the radial velocity becomes zero. Those are called the turning points.

In other words, the motion is confined such that

$$r_{min} \leq r \leq r_{max}$$

Now we have an idea about how the radial coordinate change, The motion is constrained to move bet ween two r values.

If this motion is periodic, we call it a closed orbit.

IS THE MOTION PERIODIC

Let's discuss that in detail in the next lecture