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CENTRAL FORCE MOTION

In the last class, we discussed about the equation of motion of an object in a central force field. We discussed that the total energy and the angular momentum are conserved quantities.

$$E = T + V \quad (1)$$
$$= \frac{1}{2}\mu\dot{r}^2 + + \frac{1}{2}\mu r^2\dot{\theta}^2 + V(r) \quad (2)$$
$$= \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu\dot{r}^2} + V(r) \quad (3)$$

$V(r)$ can be calculated using the type of the force field. For example, for the inverse square force law:

$$V(r) = -\int F(r)dr \quad (4)$$
$$= -\int -\frac{k}{r^2}dr \quad (5)$$
$$= -\frac{k}{r} + C \quad (6)$$

Taking the potential at the infinite separation equals zero:

$$V(r) = -\frac{k}{r} \quad (7)$$

Anyways, we can write that the total energy as:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r) \quad (8)$$

We can write this as:

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{eff} \quad (9)$$

Where the effective potential is divined as:

$$V_{eff} = \frac{l^2}{2\mu r^2} + V(r) \quad (10)$$

The first part of the effective potential is called the centrifugal potential.

Also, by looking at the eq.(8), we can write the $\dot{r}$ as:

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - V_{eff}\right)} \quad (11)$$

According to the above equation, when the goal energy equals the effective potential, $\dot{r}$ equals zero, which is called a turning point. (In another words, at the turning points, particle sees a potential barrier)

That means, the relative difference between the total energy and the effective potential energy is an important quantity in describing the orbit in a central force field.

Let's plot the effective potential.

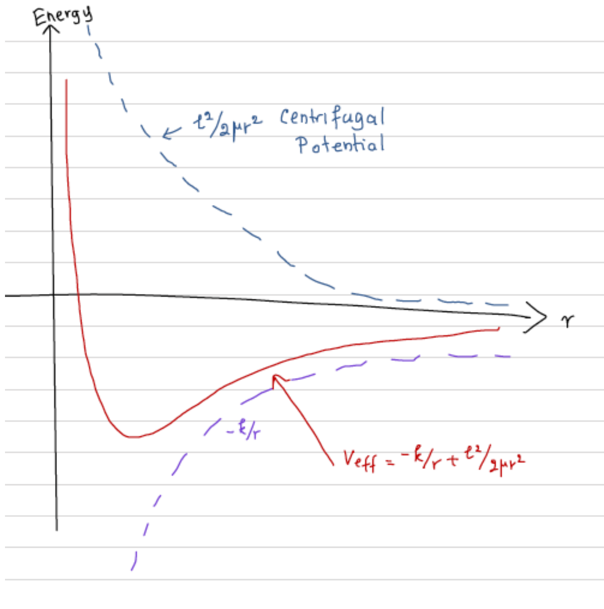

Figure 1: Effective potential

In the sketch (Figure1) of effective potential, we can identify three intersted regions

- If Total energy is larger than zero (as shown by $E_1, \quad E_1 \leq 0$)

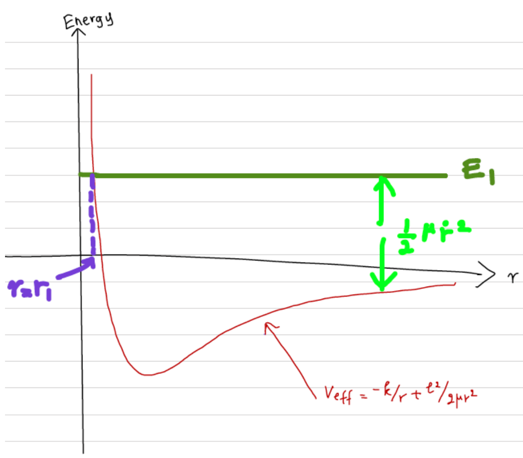

Figure 2: Case 1 $E_1 \leq 0$

We find only one turning point $r = r_1$,

The particle move towards the force center, from infinitely far away until it strikes the potential barrier at $r = r_1$, and reflected back toward infinitely larger r.

At any point r, we can find $\frac{1}{2}\mu\dot{r}^2$ as sown in the figure. We can see how the $\frac{1}{2}\mu\dot{r}^2$ changes as you change the position (one example is marked in light green). Velocity keeps changing as the position changes.

For this particular case, when the total energy is larger than zero, we see there is only one possible turning point.

- If the total energy is as shown in E_2 in the figure:

  You can see that $E - V_{eft}$ becomes zero at two possible position values r_1 and r_4. The motion of the particle is bound between r_2 and r_4.

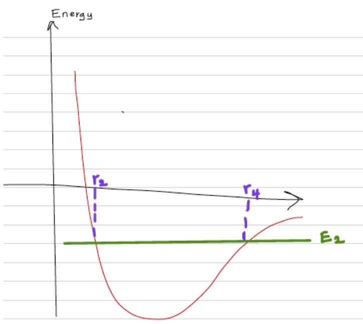

Figure 3: Case II

- If the total energy E_3 equals the minimum value of the effective potential as shown in the figure

  The motion of the particle is limited to a particular r value given by r_3 in the figure. That means a circular motion.

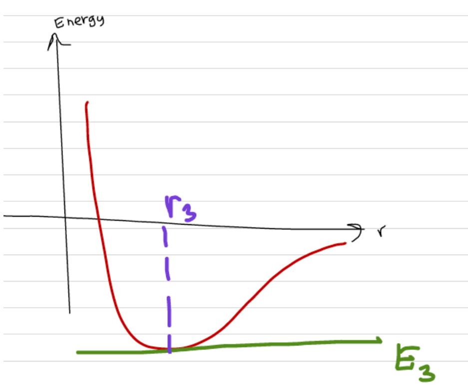

Figure 4: Case III

Now we understand, the type of the orbit largely depends on the effective potential and the total energy of the system. With this knowledge, let's move on to the planetary motion.

PLANETARY MOTION

We would like to know, how the particle moves in an inverse square law potential. Because, planetary motion is caused by the gravitational attraction, which is an inverse-square law force field.

We are interested in looking at how θ changes as a function of r.

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} \tag{12}$$

$$= \frac{\dot{\theta}}{\dot{r}} \tag{13}$$

$$d\theta = \frac{\dot{\theta}}{\dot{r}}dr \tag{14}$$

We know that

$$\dot{\theta} = \frac{l}{\mu r^2} \tag{15}$$

and

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - V_(r)\right)} \tag{16}$$

where,

$$V_{eff} = V(r) + \frac{l^2}{2\mu r^2} \tag{17}$$

$$= -\frac{k}{r} + \frac{l^2}{2\mu r^2} \tag{18}$$

By substituting them in the eq.(50),

$$\theta = \int \frac{\frac{l^2}{\mu r^2}}{\sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)}} dr + C \tag{19}$$

$$\theta = \int \frac{\frac{l^2}{r^2}}{\sqrt{2\mu\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)}} dr + C \tag{20}$$

Let's substitute, $u = \frac{1}{r}$:

$$u = \frac{1}{r} \tag{21}$$

$$du = -\frac{1}{r^2} dr \tag{22}$$

$$dr = -r^2 du \tag{23}$$

Substituting in eq. (20),we get

$$\theta = \int \frac{-l du}{\sqrt{2\mu\left(E - \frac{l^2 u^2}{2\mu} + ku\right)}} + C \tag{24}$$

$$\theta = -\int \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \frac{2\mu k}{l^2} u - u^2}} + C \tag{25}$$

In order to solve this equation,we use a standard form of integral:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{-a}} Sin^{-1}\left[\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right] + Constant \tag{26}$$

By comparing eq.(25) and (26): we have

$$a \to -1 \tag{27}$$

$$b \to \frac{2\mu k}{l^2} \tag{28}$$

$$c \to \frac{2\mu E}{l^2} \tag{29}$$

By solving the integral in eq. (25), using the standard integration:

$$
\begin{aligned}
\theta &= Sin^{-1}\left[\frac{-2u+\frac{2\mu k}{l^2}}{\sqrt{\frac{4\mu^2k^2}{l^4}+4\frac{2\mu E}{l^2}}}\right]+Constant && (30)\\
&= Sin^{-1}\left[\frac{-2u+\frac{2\mu k}{l^2}}{\sqrt{\frac{4\mu^2k^2}{l^4}+\frac{8\mu E}{l^2}}}\right]+Constant && (31)\\
\theta+C &= Sin^{-1}\left[\frac{-2u+\frac{2\mu k}{l^2}}{\sqrt{\frac{4\mu^2k^2}{l^4}+\frac{8\mu E}{l^2}}}\right] && (32)\\
Sin\,(\theta+C) &= \left[\frac{-2u+\frac{2\mu k}{l^2}}{\sqrt{\frac{4\mu^2k^2}{l^4}+\frac{8\mu E}{l^2}}}\right] && (33)
\end{aligned}
$$

Now if we define the condition that at $r_{min},\ \theta = 0$,

At r_{min}, u becomes maximum, That means $-2u$ becomes minimum. So, at $\theta = 0$, the right hand side of the above equation is a minimum. $Sinx$ is minimum, when $x \rightarrow= \pi/2$, That gives, $C = -\pi/2$ in the above equation: We can then write:

$$
Sin\,(\theta-\pi/2) = \left[\frac{-2u+\frac{2\mu k}{l^2}}{\sqrt{\frac{4\mu^2k^2}{l^4}+\frac{8\mu E}{l^2}}}\right] \tag{34}
$$

$$
-Cos\theta = \left[\frac{-2u+\frac{2\mu k}{l^2}}{\sqrt{\frac{4\mu^2k^2}{l^4}+\frac{8\mu E}{l^2}}}\right] \tag{35}
$$

which gives:

$$
Cos\theta = \left[\frac{2u-\frac{2\mu k}{l^2}}{\sqrt{\frac{4\mu^2k^2}{l^4}+\frac{8\mu E}{l^2}}}\right] \tag{36}
$$

$$
Cos\theta = \frac{\frac{2\mu k}{l^2}\left(\frac{ul^2}{\mu k}-1\right)}{\frac{2\mu k}{l^2}\left(\sqrt{1+\frac{2l^2E}{\mu k^2}}\right)} \tag{37}
$$

$$
Cos\theta = \frac{\frac{ul^2}{\mu k}-1}{\sqrt{1+\frac{2l^2E}{\mu k^2}}} \tag{38}
$$

Let's define

$$
\alpha = \frac{l^2}{\mu k} \tag{39}
$$

$$
\epsilon = \sqrt{1+\frac{2l^2E}{\mu k^2}} \tag{40}
$$

Equation (38) becomes:

$$Cos\theta = \frac{\alpha u - 1}{\epsilon} \tag{41}$$
$$\alpha u = 1 + uCos\theta \tag{42}$$
$$\frac{\alpha}{r} = 1 + \epsilon Cos\theta \tag{43}$$

Depending on the value of ϵ, the orbital shape in the inverse square law force field can be a circle, ellipse, parabola, or hyperbola.

We will talk about this in the next class.