Department of Physics, Southern Illinois University Instructor: Thushari Jayasekera (thushari@siu.edu)

Ocean Tides

What are ocean tides?

There are periodic rises and falls of water bodies around the earth. This rising of water level occur twice a day. What is the cause of this? Isaac Newton was the first one to explain this mathematically.

Basically the ocean tides are due to the different on the force on a mass placed on the different place on the earth's surface. Now we have to find out the force on a mass m placed on different place on the earth's surface.

We want to find out why the water on the surface of the earth has different forces on different times of the day. Let's first consider a mass on the surface of the earth and find out the total force acting on it. Let's look at the following figure.

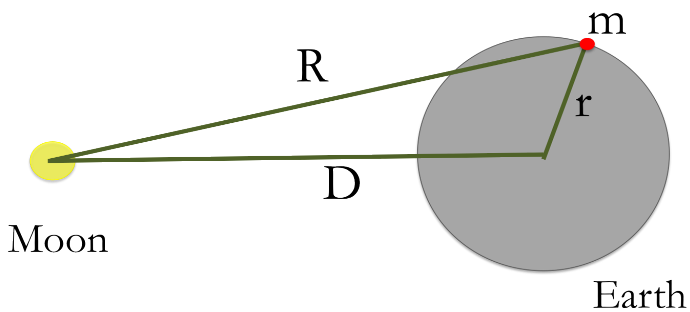

Figure 1: What is the total force on the mass m, which is placed on the surface earth?

What is the total force on m? This can be a complicated problem, as earth is rotating and moon is orbiting around the earth. Non of these are really inertial frames. We need to look at the situation from an inertial frame. Let's look at the figure as follows.

So we set up an inertial frame as shown in the following figure, and measure quantities relative to the inertial frame, such that Newton's laws can be applied.

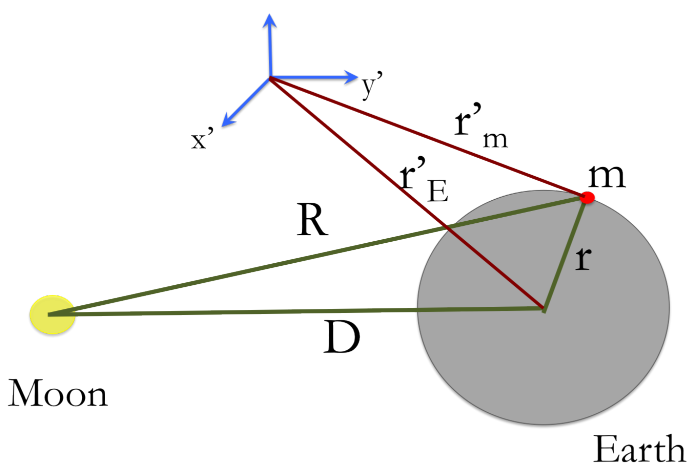

Figure 2: We measure all the quatities relative town inertial frame

We set up an inertial frame $x'y'z'$ as shown in the figure.

As measure from the inertial frame:, the force on m due to the earth and the moon is:

$$m\ddot{r}'_m = -G\frac{mM_E}{r^2}\vec{e}_r - G\frac{mM_m}{R^2}\vec{e}_R \tag{1}$$

If I can find out the $\ddot{r}$,then I can compare the force on m at different position on the earth. (We could not directly write $\ddot{r}$ because, the quantity r is measured from the non-inertial frames.

$$\ddot{r} = \ddot{r}'_m - \ddot{r}'_E \tag{2}$$

We also need to find an equation for $\ddot{r}'_E$, Let's write the force on the Center of mass on the earth due to the moon, as measured from the inertial frame of reference:

$$M_E\ddot{r}'_E = -G\frac{M_E M_m}{D^2}\vec{e}_D \tag{3}$$

Now we can combine the equations (1), (2), and (3) as:

$$\begin{aligned}
\ddot{r} &= \ddot{r}'_m - \ddot{r}'_E \qquad (4)\\
&= \frac{m\ddot{r}'_m}{m} - \frac{M_E r'_E}{M_E}\\
&= -G\frac{M_E}{r^2}\vec{e}_r - G\frac{M_m}{R^2}\vec{e}_R + G\frac{M_m}{D^2}\vec{e}_D\\
&= -G\frac{M_E}{r^2}\vec{e}_r - GM_m\left[\frac{\vec{e}_R}{R^2} - \frac{\vec{e}_D}{D^2}\right] \qquad (5)
\end{aligned}$$

It is clear that the first part of the eq.(5) comes due to the attraction of m towards the center of the earth. The second term is called the **Tidal Force**. By looking at the equation, we can see that, the tidal force result in from the difference of the attractive force on the center of the earth and the surface of the earth due to the moon.

Also, as the tidal force depends on D and R, the tidal force highly depends on where we sit on the earth with relative to the moon.

Let's look at the tidal force at different positions relative to the moon.

Possible Tidal forces at different positions on the surface of the Earth relative to the Moon

Case I: Position A

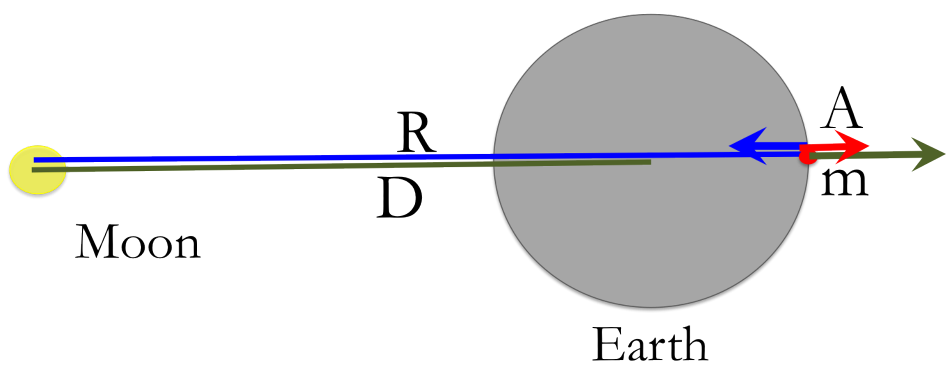

Figure 3: The resultant tidal force is marked in red

As shown in the figure, both $\vec{e}_R$ and $\vec{e}_D$ are pointing in the same direction. I can write $R = r + D$ where r is the radius of the earth.

So the tidal free F_T can be written as:

$$F_T = -GmM_m\left[\frac{1}{(D+r)^2} - \frac{1}{D^2}\right] \tag{6}$$

$$F_T = -\frac{GmM_m}{D^2}\left[\frac{1}{(1+\frac{r}{D})^2} - 1\right] \tag{7}$$

We can expand the first term in the brackets as $(1+x)^{-2} = 1 - 2x + 3x^2 + \cdots$, for small x.

$$F_T = -\frac{GmM_m}{D^2}\left[1 - \frac{2r}{D} + 3(\frac{r}{D})^2 + \cdots\right] \tag{8}$$

Only considering to the lowest order, we have

$$F_T = 2\frac{GmM_m r}{D^3} \tag{9}$$

Case II: Position B

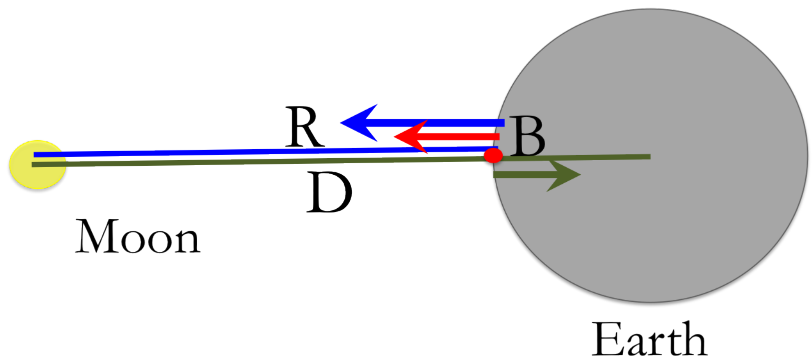

Figure 4: The resultant tidal force is marked in red

We can calculate the tidal force at the point B as shown in the above figure.

$$F_T = -GmM_m\left[\frac{1}{(D-r)^2} - \frac{1}{D^2}\right] \tag{10}$$

We can carry out the same math, and at the end, we will have an equation: Only considering to the lowest order, we have

$$F_T = -2\frac{GmM_m r}{D^3} \tag{11}$$

The magnitude of the tidal force is as same as at point A, however, now it is pointing in the $-\hat{x}$ direction.

Case II: Position C

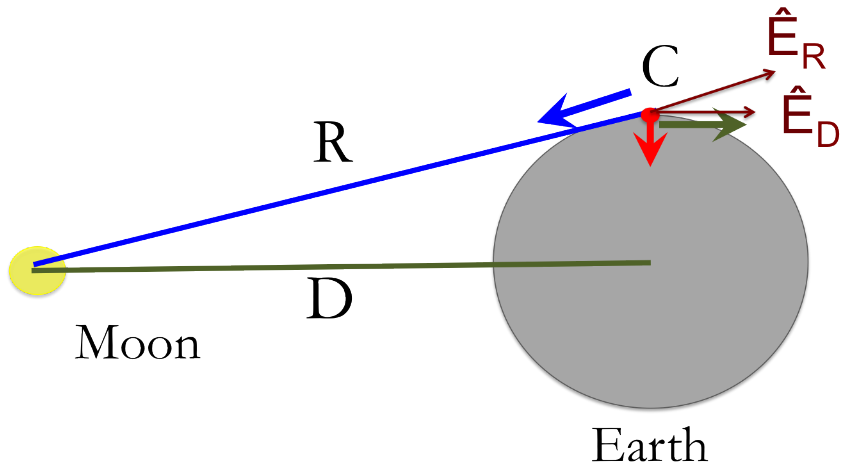

Figure 5: The resultant tidal force is marked in red

Starting from the equation (5),we can evaluate an expression for the tidal force at the Point C.

$$F_T = -GmM_m \left[\frac{\vec{e}_R}{R^2} - \frac{\vec{e}_D}{D^2} \right] \tag{12}$$

In the previous two cases, both $\vec{e}_R$ and $\vec{e}_D$ are along the $\hat{x}$ axis. However, now it is not the case. The directions $\vec{e}_R$ and $\vec{e}_D$ are shown in the figure. If we think about the $\hat{x}$ component of the force in $\vec{e}_R$ direction and the force in $\vec{e}_D$ direction cancel each other. SO we end up with a force component in the negative $\hat{y}$ direction.

By taking the y component as:

$$F_T = -GmM_m \left[\frac{\vec{e}_R}{R^2} \times \frac{r}{R} \right] \tag{13}$$

and then approximating $R\ D$, we get,

$$F_T = -GmM_m \frac{r}{D^3} \tag{14}$$

The resultant tidal force at point C is in the negative $\hat{y}$ direction. In the similar fashion, we can find the tidal force at a point just opposite to the point C is in $+\hat{y}$ direction.

This is the over all tidal picture:

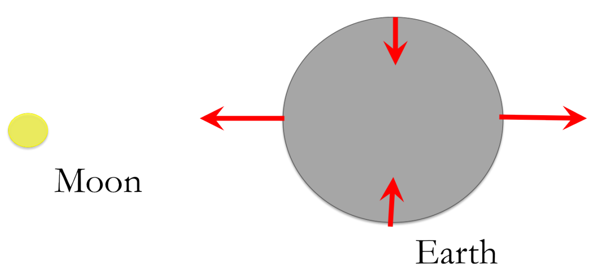

Figure 6: The resultant tidal forces

Notice that this gives a model for the Ocean tides, we did not consider the motion of the Earth around the sun, etc

Anyway, within this model, let's calculate the height of the maximum tides.