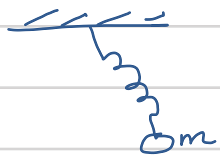
Review-2

Q-1) A spring pendulum consists of a mass m attached to one end of a massless spring with spring constant k. The other end of the spring is tied to a fixed support. When no weight is on the spring, its length is ℓ. Assume that the motion of the system is confined to a vertical plane. Derive the equation of motion. Solve the equation of motion in the approximation of small angular and radial displacements from equilibrium.



Q-2) A sphere of mass M radius R rolls without slipping down a triangular block of mass m that is free to move on a frictionless horizontal surface.

(a) Find the Lagrangean and state the Lagrange's Equation of motion for this system, which is subjected to the force of gravity.

(b) Find the motion of the system by integrating Lagrange's equation given that all objects are initially at rest, and sphere's center is at a distance H above the surface.

Q-3) Consider a particle of mass m moving in the force field.

$$F(r) = -\frac{k}{r^2} + \frac{k'}{r^3} \qquad k>0$$

(a) Considering what we discussed in central force motion, how many coordinates are required to explain the motion.

(b) What is the Lagrangean for this motion.

(c) What is the Hamiltonian.

(d) Set up the Euler Lagrangean Equation.

(e) Do you identify any constants of motion.

$\boxed{Q\text{-}4}$ A mechanical system known as Atwood's machine consists of two weights m_1 & m_2 respectively, connected by a light inextensible cord of length l, which passes over a pulley. Pulley has a mass M. Find out the equation of motion.
What is the condition to have the system in equilibrium?
Find the tension of the cord for the case where the pulley is massless.

Example 7.7

Example 7.12.

c

Since we need to keep track of the length of the spring plane polar coordinates will be a better choice.

You can start writing this as $x = r\sin\theta \rightarrow \dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$
$y = r\cos\theta \rightarrow \dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$$v^2 = \dot{x}^2 + \dot{y}^2$$
$$= (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 + (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2$$
$$= \dot{r}^2 + r^2\dot{\theta}^2$$

One could start from here.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = -mgr\cos\theta + \frac{1}{2}k(r-\ell)^2$$

$$L = T - V = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{1}{2}k(r-\ell)^2 + mgr\cos\theta$$

We have to set up two Euler Lagrangean Equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - k(r-\ell) + mg\cos\theta$$

$$m\ddot{r} - \left[mr\dot{\theta}^2 - k(r-\ell) + mg\cos\theta\right] = 0 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left[mr^2\dot{\theta}\right] + mgr\cos\theta = 0$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\theta = 0$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0 \quad \text{— (2)}$$

At equilibrium $\dot{r} = 0 \quad \dot{\theta} = 0$
$\ddot{r} = 0 \quad \ddot{\theta} = 0$

(2) $\Rightarrow$ $\sin\theta_0 = 0$

$\theta_0 = 0 \leftarrow$ Equilibrium Position

(1) $\Rightarrow$

$$+k(r-\ell) - mg\cos\theta_0 = 0$$
$$k(r-\ell) = mg$$
$$r_0 = \ell + \frac{mg}{k}$$

Now we look for small angles. let $r = r_0 + \rho$ $\quad \dot{r} = \dot{\rho} \quad \ddot{r} = \ddot{\rho}$

$$m\ddot{r} - mr\dot{\theta}^2 + k(r-\ell) - mg = 0$$

$$m\ddot{\rho} - m(r_0+\rho)\dot{\theta}^2 + k(r_0+\rho-\ell) - mg = 0$$

$$m\ddot{\rho} - m(r_0+\rho)\dot{\theta}^2 + k\rho = 0 \quad \& \quad (r_0+\rho)\ddot{\theta} + 2\dot{\rho}\dot{\theta} + g\theta = 0$$

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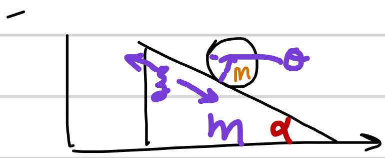


Coordinates of m

$$x_m = x \qquad y_m = 0$$

Coordinates for M

$$x_M = x + \xi\cos\alpha$$
$$y = H - \xi\sin\alpha$$

Since the sphere is moving without slipping. $\xi = R\theta$

So we can use x and θ or

x and ξ as the generalized coordinates.

$$T_M = \frac{1}{2}M\left[(\dot{x} + \dot{\xi}\cos\alpha)^2 + (-\dot{\xi}\cos\alpha)^2\right]$$
$$= \frac{1}{2}M\left[\dot{x}^2 + 2\dot{x}\dot{\xi}\cos\alpha + \dot{\xi}^2\right]$$

$$T_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\frac{2}{5}MR^2\dot{\theta}^2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{5}MR^2\dot{\theta}^2$$
$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{5}MR^2\frac{\dot{\xi}^2}{R^2}$$
$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{5}M\dot{\xi}^2$$

$$V_m = Mg(H - R\theta\sin\alpha)$$

$$L = \frac{1}{2}M\left[\dot{x}^2 + 2\dot{x}\dot{\xi}\cos\alpha + \dot{\xi}^2\right] + \frac{1}{2}m\dot{x}^2 + \frac{1}{5}m\dot{\xi}^2 - Mg(H - \cancel{R\theta}\,\xi\sin\alpha)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial \dot{x}} = M\dot{x} + M\dot{\xi}\cos\alpha + m\dot{x}$$
$$= (M+m)\dot{x} + M\dot{\xi}\cos\alpha$$

$$(M+m)\ddot{x} + m\ddot{\xi}\cos\alpha = 0 \quad \text{——— } ①$$

$$\frac{\partial L}{\partial \xi} = mg\sin\alpha.$$

$$\frac{\partial L}{\partial \dot{\xi}} = M\dot{x}\cos\alpha + M\dot{\xi} + \frac{2}{5}m\dot{\xi}$$
$$= \left(M + \frac{2}{5}m\right)\dot{\xi} + M\dot{x}\cos\alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = \left(M + \frac{2}{5}m\right)\ddot{\xi} + M\ddot{x}\cos\alpha$$

You can combine eq ① & ② & then integrate.

$$\underset{7/5\,m}{(\cancel{M+m})}\,\ddot{\xi} + M\ddot{x}\cos\alpha - Mg\,\cancel{\xi}\sin\alpha = 0$$

$$\frac{7}{5}m\ddot{\xi} + M\ddot{x}\cos\alpha - Mg\sin\alpha = 0 \quad \text{——— } ②$$

Q-3

$$F(r) = \frac{-k}{r^2} + \frac{k'}{r^3}$$

$$V(r) = -\int_{\infty}^{r} F(r)\,dr = -\left[\frac{-k(r)^{-1}}{-1} + \frac{k'(r)^{-2}}{-2}\right]_{\infty}^{r}$$

$$= -\left[\frac{k}{r} - \frac{k'}{2r^2}\right]_{\infty}^{r}$$

$$= \frac{-k}{r} + \frac{k'}{2r^2}$$

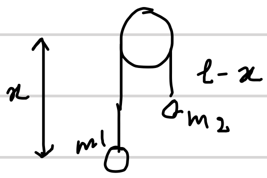
This is a central force field. Angular momentum is conserved. And the force is in the direction $\hat{r}$.

So the motion should be constrained to a plane.

(r, θ) will be generalized coordinates.

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - \left[\frac{-k}{r} + \frac{k'}{2r^2}\right]$$

Q-4 In the first approach, we will be using the constraints to reduce the number of coordinates.



$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$a\theta = l_0 + x$$

$$a\dot{\theta} = \dot{x}$$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I \frac{\dot{x}^2}{a^2}$$

$$V = -m_1 g x - m_2 g (l - x)$$

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}^2 + (m_1 - m_2) g x - m_2 g l$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\left(m_1 + m_2 + \frac{I}{a}\right)\ddot{x} + (m_1 - m_2) g = 0$$

$$\ddot{x} = \frac{(m_1 - m_2) g}{(m_1 + m_2 + I/a^2)}$$

We can find the Eq^m condition by making $\ddot{x} = 0$

In this problem we can also easily apply $F = ma$

$$m_1 \ddot{x} = m_1 g - T$$

$$T = -m_1 \frac{(m_1 - m_2) g}{(m_1 + m_2 + I/a^2)} + m_1 g$$

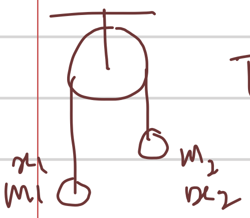
$$I = \frac{m a^2}{4}$$

$$T = \frac{-m_1^2 g + m_1 m_2 g + m_1^2 g + m_1 m_2 g + I/a^2 \, m_1 g}{(m_1 + m_2 + I/a^2)}$$

$$= \frac{2 m_1 m_2 g}{(m_1 + m_2 + I/a^2)}$$

(b)

We can use the Lagrangean Undetermined Multiplier Method



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \qquad V = -m_1 g x_1 - m_2 g x_2$$

Constraint

$$x_1 + x_2 = \ell \qquad g = x_1 + x_2 - \ell = 0$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + m_1 g x_1 + m_2 g x_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0$$

$$m_1 \ddot{x}_1 - m_1 g + \lambda = 0$$

$$m_2 \ddot{x}_2 - m_2 g + \lambda = 0$$

$$\ddot{x}_1 - g + \lambda / m_1 = 0$$

$$\ddot{x}_1 - g + \lambda / m_2 = 0$$

$$(\ddot{x}_1 + \ddot{x}_2) - 2g + \frac{\lambda}{m_1} + \frac{\lambda}{m_2} = 0$$

$$2g = \lambda \left[\frac{1}{m_1} + \frac{1}{m_2}\right]$$

$$\lambda = \frac{2g\, m_1 m_2}{m_1 + m_2}$$

Constraint force $\Phi_{x_1} = \lambda$

You can compare it with the previous problem.