Metals occupy a rather special position in the study of solids, sharing a variety of striking properties that other solids (such as quartz, sulfur, or common salt) lack. They are excellent conductors of heat and electricity, are ductile and malleable, and display a striking luster on freshly exposed surfaces. The challenge of accounting for these metallic features gave the starting impetus to the modern theory of solids.

Although the majority of commonly encountered solids are nonmetallic, metals have continued to play a prominent role in the theory of solids from the late nineteenth century to the present day. Indeed, the metallic state has proved to be one of the great fundamental states of matter. The elements, for example, definitely favor the metallic state: over two thirds are metals. Even to understand nonmetals one must also understand metals, for in explaining why copper conducts so well, one begins to learn why common salt does not.

During the last hundred years physicists have tried to construct simple models of the metallic state that account in a qualitative, and even quantitative, way for the characteristic metallic properties. In the course of this search brilliant successes have appeared hand in hand with apparently hopeless failures, time and again. Even the earliest models, though strikingly wrong in some respects, remain, when properly used, of immense value to solid state physicists today.

In this chapter we shall examine the theory of metallic conduction put forth by P. Drude[1] at the turn of the century. The successes of the Drude model were considerable, and it is still used today as a quick practical way to form simple pictures and rough estimates of properties whose more precise comprehension may require analysis of considerable complexity. The failures of the Drude model to account for some experiments, and the conceptual puzzles it raised, defined the problems with which the theory of metals was to grapple over the next quarter century. These found their resolution only in the rich and subtle structure of the quantum theory of solids.

## BASIC ASSUMPTIONS OF THE DRUDE MODEL

J. J. Thomson's discovery of the electron in 1897 had a vast and immediate impact on theories of the structure of matter, and suggested an obvious mechanism for conduction in metals. Three years after Thomson's discovery Drude constructed his theory of electrical and thermal conduction by applying the highly successful kinetic theory of gases to a metal, considered as a gas of electrons.

In its simplest form kinetic theory treats the molecules of a gas as identical solid spheres, which move in straight lines until they collide with one another.[2] The time taken up by a single collision is assumed to be negligible, and, except for the forces coming momentarily into play during each collision, no other forces are assumed to act between the particles.

Although there is only one kind of particle present in the simplest gases, in a metal there must be at least two, for the electrons are negatively charged, yet the metal is electrically neutral. Drude assumed that the compensating positive charge was at-

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[1] *Annalen der Physik* **1**, 566 and **3**, 369 (1900).

[2] Or with the walls of the vessel containing them, a possibility generally ignored in discussing metals unless one is interested in very fine wires, thin sheets, or effects at the surface.

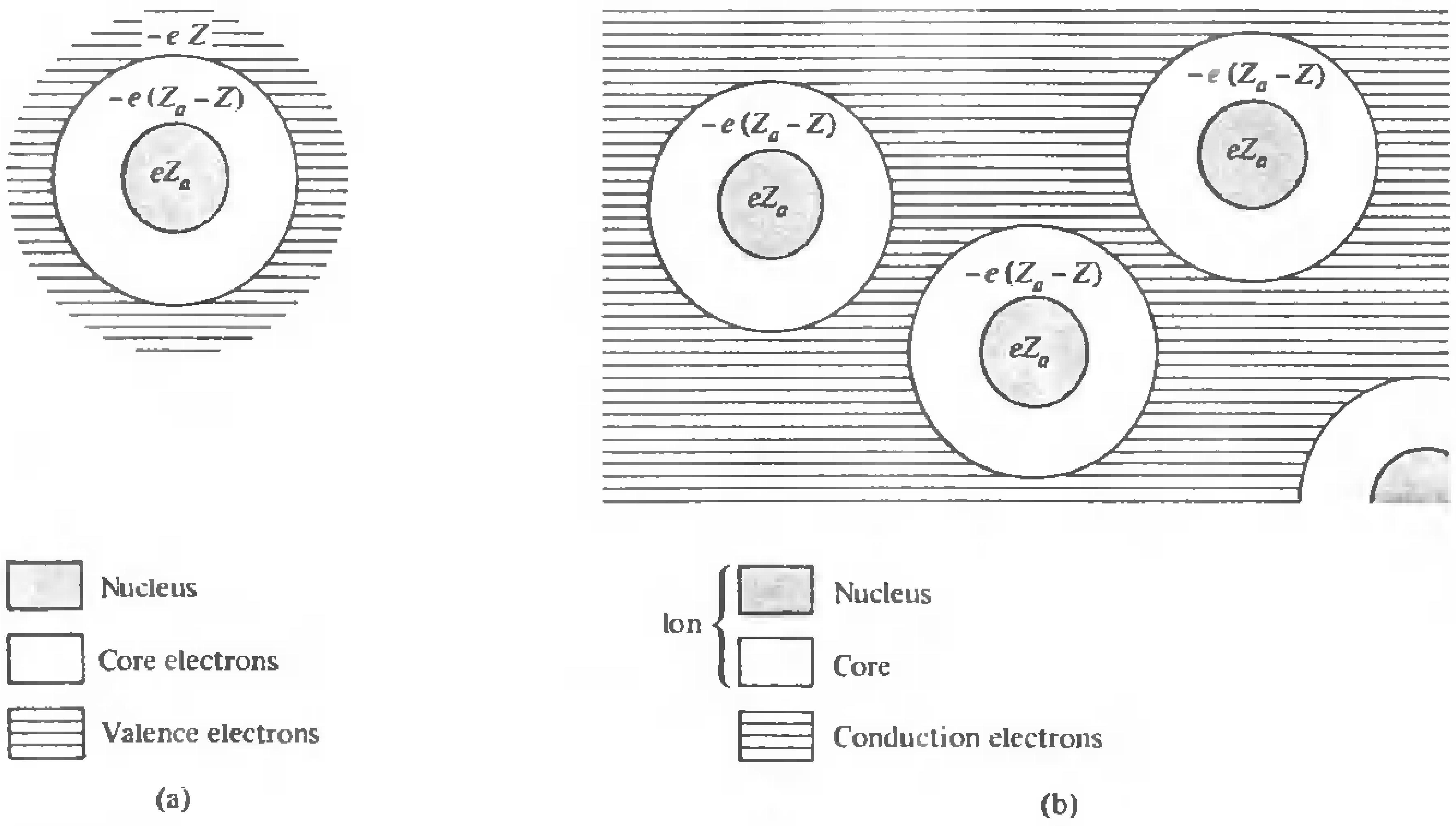

**Figure 1.1**
(a) Schematic picture of an isolated atom (not to scale). (b) In a metal the nucleus and ion core retain their configuration in the free atom, but the valence electrons leave the atom to form the electron gas.

tached to much heavier particles, which he considered to be immobile. At his time, however, there was no precise notion of the origin of the light, mobile electrons and the heavier, immobile, positively charged particles. The solution to this problem is one of the fundamental achievements of the modern quantum theory of solids. In this discussion of the Drude model, however, we shall simply assume (and in many metals this assumption can be justified) that when atoms of a metallic element are brought together to form a metal, the valence electrons become detached and wander freely through the metal, while the metallic ions remain intact and play the role of the immobile positive particles in Drude's theory. This model is indicated schematically in Figure 1.1. A single isolated atom of the metallic element has a nucleus of charge $eZ_a$, where $Z_a$ is the atomic number and $e$ is the magnitude of the electronic charge[3]: $e = 4.80 \times 10^{-10}$ electrostatic units (esu) $= 1.60 \times 10^{-19}$ coulombs. Surrounding the nucleus are $Z_a$ electrons of total charge $-eZ_a$. A few of these, $Z$, are the relatively weakly bound valence electrons. The remaining $Z_a - Z$ electrons are relatively tightly bound to the nucleus, play much less of a role in chemical reactions, and are known as the core electrons. When these isolated atoms condense to form a metal, the core electrons remain bound to the nucleus to form the metallic ion, but the valence electrons are allowed to wander far away from their parent atoms. In the metallic context they are called conduction electrons.[4]

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[3] We shall always take $e$ to be a positive number.

[4] When, as in the Drude model, the core electrons play a passive role and the ion acts as an indivisible inert entity, one often refers to the conduction electrons simply as "the electrons," saving the full term for times when the distinction between conduction and core electrons is to be emphasized.

Drude applied kinetic theory to this "gas" of conduction electrons of mass $m$, which (in contrast to the molecules of an ordinary gas) move against a background of heavy immobile ions. The density of the electron gas can be calculated as follows:

A metallic element contains $0.6022 \times 10^{24}$ atoms per mole (Avogadro's number) and $\rho_m/A$ moles per cm$^3$, where $\rho_m$ is the mass density (in grams per cubic centimeter) and $A$ is the atomic mass of the element. Since each atom contributes $Z$ electrons, the number of electrons per cubic centimeter, $n = N/V$, is

$$n = 0.6022 \times 10^{24} \frac{Z\rho_m}{A}. \tag{1.1}$$

Table 1.1 shows the conduction electron densities for some selected metals. They are typically of order $10^{22}$ conduction electrons per cubic centimeter, varying from $0.91 \times 10^{22}$ for cesium up to $24.7 \times 10^{22}$ for beryllium.[5] Also listed in Table 1.1 is a widely used measure of the electronic density, $r_s$, defined as the radius of a sphere whose volume is equal to the volume per conduction electron. Thus

$$\frac{V}{N} = \frac{1}{n} = \frac{4\pi r_s^{\,3}}{3}; \quad r_s = \left(\frac{3}{4\pi n}\right)^{1/3}. \tag{1.2}$$

Table 1.1 lists $r_s$ both in angstroms ($10^{-8}$ cm) and in units of the Bohr radius $a_0 = \hbar^2/me^2 = 0.529 \times 10^{-8}$ cm; the latter length, being a measure of the radius of a hydrogen atom in its ground state, is often used as a scale for measuring atomic distances. Note that $r_s/a_0$ is between 2 and 3 in most cases, although it ranges between 3 and 6 in the alkali metals (and can be as large as 10 in some metallic compounds).

These densities are typically a thousand times greater than those of a classical gas at normal temperatures and pressures. In spite of this and in spite of the strong electron-electron and electron-ion electromagnetic interactions, the Drude model boldly treats the dense metallic electron gas by the methods of the kinetic theory of a neutral dilute gas, with only slight modifications. The basic assumptions are these:

1. Between collisions the interaction of a given electron, both with the others and with the ions, is neglected. Thus in the absence of externally applied electromagnetic fields each electron is taken to move uniformly in a straight line. In the presence of externally applied fields each electron is taken to move as determined by Newton's laws of motion in the presence of those external fields, but neglecting the additional complicated fields produced by the other electrons and ions.[6] The neglect of electron-electron interactions between collisions is known as the *independent electron approximation.* The corresponding neglect of electron-ion interactions is known as the *free electron approximation.* We shall find in subsequent chapters that

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[5] This is the range for metallic elements under normal conditions. Higher densities can be attained by application of pressure (which tends to favor the metallic state). Lower densities are found in compounds.

[6] Strictly speaking, the electron-ion interaction is not entirely ignored, for the Drude model implicitly assumes that the electrons are confined to the interior of the metal. Evidently this confinement is brought about by their attraction to the positively charged ions. Gross effects of the electron-ion and electron-electron interaction like this are often taken into account by adding to the external fields a suitably defined internal field representing the average effect of the electron-electron and electron-ion interactions.

Table 1.1
**FREE ELECTRON DENSITIES OF SELECTED METALLIC ELEMENTS**[a]

| ELEMENT | $Z$ | $n$ ($10^{22}$/cm$^3$) | $r_s$(Å) | $r_s/a_0$ |
|---|---|---|---|---|
| Li (78 K) | 1 | 4.70 | 1.72 | 3.25 |
| Na (5 K) | 1 | 2.65 | 2.08 | 3.93 |
| K (5 K) | 1 | 1.40 | 2.57 | 4.86 |
| Rb (5 K) | 1 | 1.15 | 2.75 | 5.20 |
| Cs (5 K) | 1 | 0.91 | 2.98 | 5.62 |
| Cu | 1 | 8.47 | 1.41 | 2.67 |
| Ag | 1 | 5.86 | 1.60 | 3.02 |
| Au | 1 | 5.90 | 1.59 | 3.01 |
| Be | 2 | 24.7 | 0.99 | 1.87 |
| Mg | 2 | 8.61 | 1.41 | 2.66 |
| Ca | 2 | 4.61 | 1.73 | 3.27 |
| Sr | 2 | 3.55 | 1.89 | 3.57 |
| Ba | 2 | 3.15 | 1.96 | 3.71 |
| Nb | 1 | 5.56 | 1.63 | 3.07 |
| Fe | 2 | 17.0 | 1.12 | 2.12 |
| Mn (α) | 2 | 16.5 | 1.13 | 2.14 |
| Zn | 2 | 13.2 | 1.22 | 2.30 |
| Cd | 2 | 9.27 | 1.37 | 2.59 |
| Hg (78 K) | 2 | 8.65 | 1.40 | 2.65 |
| Al | 3 | 18.1 | 1.10 | 2.07 |
| Ga | 3 | 15.4 | 1.16 | 2.19 |
| In | 3 | 11.5 | 1.27 | 2.41 |
| Tl | 3 | 10.5 | 1.31 | 2.48 |
| Sn | 4 | 14.8 | 1.17 | 2.22 |
| Pb | 4 | 13.2 | 1.22 | 2.30 |
| Bi | 5 | 14.1 | 1.19 | 2.25 |
| Sb | 5 | 16.5 | 1.13 | 2.14 |

[a] At room temperature (about 300 K) and atmospheric pressure, unless otherwise noted. The radius $r_s$ of the free electron sphere is defined in Eq. (1.2). We have arbitrarily selected one value of $Z$ for those elements that display more than one chemical valence. The Drude model gives no theoretical basis for the choice. Values of $n$ are based on data from R. W. G. Wyckoff, *Crystal Structures*, 2nd ed., Interscience, New York, 1963.

although the independent electron approximation is in many contexts surprisingly good, the free electron approximation must be abandoned if one is to arrive at even a qualitative understanding of much of metallic behavior.

**2.** Collisions in the Drude model, as in kinetic theory, are instantaneous events that abruptly alter the velocity of an electron. Drude attributed them to the electrons bouncing off the impenetrable ion cores (rather than to electron-electron collisions, the analogue of the predominant collision mechanism in an ordinary gas). We shall find later that electron-electron scattering is indeed one of the least important of the several scattering mechanisms in a metal, except under unusual conditions. However,

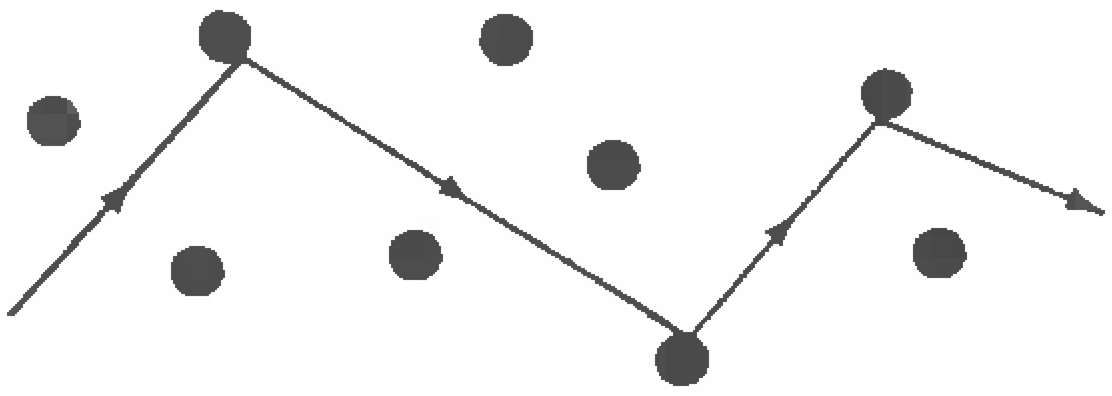

**Figure 1.2**
Trajectory of a conduction electron scattering off the ions, according to the naive picture of Drude.

the simple mechanical picture (Figure 1.2) of an electron bumping along from ion to ion is very far off the mark.[7] Fortunately, this does not matter for many purposes: a qualitative (and often a quantitative) understanding of metallic conduction can be achieved by simply assuming that there is *some* scattering mechanism, without inquiring too closely into just what that mechanism might be. By appealing, in our analysis, to only a few general effects of the collision process, we can avoid committing ourselves to any specific picture of how electron scattering actually takes place. These broad features are described in the following two assumptions.

**3.** We shall assume that an electron experiences a collision (i.e., suffers an abrupt change in its velocity) with a probability per unit time $1/\tau$. We mean by this that the probability of an electron undergoing a collision in any infinitesimal time interval of length $dt$ is just $dt/\tau$. The time $\tau$ is variously known as the relaxation time, the collision time, or the mean free time, and it plays a fundamental role in the theory of metallic conduction. It follows from this assumption that an electron picked at random at a given moment will, on the average, travel for a time $\tau$ before its next collision, and will, on the average, have been traveling for a time $\tau$ since its last collision.[8] In the simplest applications of the Drude model the collision time $\tau$ is taken to be independent of an electron's position and velocity. We shall see later that this turns out to be a surprisingly good assumption for many (but by no means all) applications.

**4.** Electrons are assumed to achieve thermal equilibrium with their surroundings only through collisions.[9] These collisions are assumed to maintain local thermodynamic equilibrium in a particularly simple way: immediately after each collision an electron is taken to emerge with a velocity that is not related to its velocity just before the collision, but randomly directed and with a speed appropriate to the temperature prevailing at the place where the collision occurred. Thus the hotter the region in which a collision occurs, the faster a typical electron will emerge from the collision.

In the rest of this chapter we shall illustrate these notions through their most important applications, noting the extent to which they succeed or fail to describe the observed phenomena.

## DC ELECTRICAL CONDUCTIVITY OF A METAL

According to *Ohm's law*, the current $I$ flowing in a wire is proportional to the potential drop $V$ along the wire: $V = IR$, where $R$, the resistance of the wire, depends on its

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[7] For some time people were led into difficult but irrelevant problems connected with the proper aiming of an electron at an ion in each collision. So literal an interpretation of Figure 1.2 is strenuously to be avoided.

[8] See Problem 1.

[9] Given the free and independent electron approximation, this is the only possible mechanism left.

dimensions, but is independent of the size of the current or potential drop. The Drude model accounts for this behavior and provides an estimate of the size of the resistance.

One generally eliminates the dependence of $R$ on the shape of the wire by introducing a quantity characteristic only of the metal of which the wire is composed. The resistivity $\rho$ is defined to be the proportionality constant between the electric field $\mathbf{E}$ at a point in the metal and the current density $\mathbf{j}$ that it induces[10]:

$$\mathbf{E} = \rho \mathbf{j}. \tag{1.3}$$

The current density $\mathbf{j}$ is a vector, parallel to the flow of charge, whose magnitude is the amount of charge per unit time crossing a unit area perpendicular to the flow. Thus if a uniform current $I$ flows through a wire of length $L$ and cross-sectional area $A$, the current density will be $j = I/A$. Since the potential drop along the wire will be $V = EL$, Eq. (1.3) gives $V = I\rho L/A$, and hence $R = \rho L/A$.

If $n$ electrons per unit volume all move with velocity $\mathbf{v}$, then the current density they give rise to will be parallel to $\mathbf{v}$. Furthermore, in a time $dt$ the electrons will advance by a distance $v\,dt$ in the direction of $\mathbf{v}$, so that $n(v\,dt)A$ electrons will cross an area $A$ perpendicular to the direction of flow. Since each electron carries a charge $-e$, the charge crossing $A$ in the time $dt$ will be $-nevA\,dt$, and hence the current density is

$$\mathbf{j} = -ne\mathbf{v}. \tag{1.4}$$

At any point in a metal, electrons are always moving in a variety of directions with a variety of thermal energies. The net current density is thus given by (1.4), where $\mathbf{v}$ is the average electronic velocity. In the absence of an electric field, electrons are as likely to be moving in any one direction as in any other, $\mathbf{v}$ averages to zero, and, as expected, there is no net electric current density. In the presence of a field $\mathbf{E}$, however, there will be a mean electronic velocity directed opposite to the field (the electronic charge being negative), which we can compute as follows:

Consider a typical electron at time zero. Let $t$ be the time elapsed since its last collision. Its velocity at time zero will be its velocity $\mathbf{v}_0$ immediately after that collision plus the additional velocity $-e\mathbf{E}t/m$ it has subsequently acquired. Since we assume that an electron emerges from a collision in a random direction, there will be no contribution from $\mathbf{v}_0$ to the average electronic velocity, which must therefore be given entirely by the average of $-e\mathbf{E}t/m$. However, the average of $t$ is the relaxation time $\tau$. Therefore

$$\mathbf{v}_{\text{avg}} = -\frac{e\mathbf{E}\tau}{m}; \quad \mathbf{j} = \left(\frac{ne^2\tau}{m}\right)\mathbf{E}. \tag{1.5}$$

This result is usually stated in terms of the inverse of the resistivity, the conductivity $\sigma = 1/\rho$:

$$\boxed{\mathbf{j} = \sigma\mathbf{E}; \quad \sigma = \frac{ne^2\tau}{m}.} \tag{1.6}$$

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[10] In general, $\mathbf{E}$ and $\mathbf{j}$ need not be parallel. One then defines a resistivity *tensor*. See Chapters 12 and 13.

This establishes the linear dependence of **j** on **E** and gives an estimate of the conductivity $\sigma$ in terms of quantities that are all known except for the relaxation time $\tau$. We may therefore use (1.6) and the observed resistivities to estimate the size of the relaxation time:

$$\tau = \frac{m}{\rho n e^2}. \tag{1.7}$$

Table 1.2 gives the resistivities of several representative metals at several temperatures. Note the strong temperature dependence. At room temperature the resistivity is roughly linear in $T$, but it falls away much more steeply as low temperatures are

Table 1.2
**ELECTRICAL RESISTIVITIES OF SELECTED ELEMENTS**[a]

| ELEMENT | 77 K | 273 K | 373 K | $\frac{(\rho/T)_{373\text{ K}}}{(\rho/T)_{273\text{ K}}}$ |
|---|---|---|---|---|
| Li | 1.04 | 8.55 | 12.4 | 1.06 |
| Na | 0.8 | 4.2 | Melted | |
| K | 1.38 | 6.1 | Melted | |
| Rb | 2.2 | 11.0 | Melted | |
| Cs | 4.5 | 18.8 | Melted | |
| Cu | 0.2 | 1.56 | 2.24 | 1.05 |
| Ag | 0.3 | 1.51 | 2.13 | 1.03 |
| Au | 0.5 | 2.04 | 2.84 | 1.02 |
| Be | | 2.8 | 5.3 | 1.39 |
| Mg | 0.62 | 3.9 | 5.6 | 1.05 |
| Ca | | 3.43 | 5.0 | 1.07 |
| Sr | 7 | 23 | | |
| Ba | 17 | 60 | | |
| Nb | 3.0 | 15.2 | 19.2 | 0.92 |
| Fe | 0.66 | 8.9 | 14.7 | 1.21 |
| Zn | 1.1 | 5.5 | 7.8 | 1.04 |
| Cd | 1.6 | 6.8 | | |
| Hg | 5.8 | Melted | Melted | |
| Al | 0.3 | 2.45 | 3.55 | 1.06 |
| Ga | 2.75 | 13.6 | Melted | |
| In | 1.8 | 8.0 | 12.1 | 1.11 |
| Tl | 3.7 | 15 | 22.8 | 1.11 |
| Sn | 2.1 | 10.6 | 15.8 | 1.09 |
| Pb | 4.7 | 19.0 | 27.0 | 1.04 |
| Bi | 35 | 107 | 156 | 1.07 |
| Sb | 8 | 39 | 59 | 1.11 |

[a] Resistivities in microhm centimeters are given at 77 K (the boiling point of liquid nitrogen at atmospheric pressure), 273 K, and 373 K. The last column gives the ratio of $\rho/T$ at 373 K and 273 K to display the approximate linear temperature dependence of the resistivity near room temperature.

Source: G. W. C. Kaye and T. H. Laby, *Table of Physical and Chemical Constants*, Longmans Green, London, 1966.

reached. Room temperature resistivities are typically of the order of microhm centimeters ($\mu$ohm-cm) or, in atomic units, of order $10^{-18}$ statohm-cm.[11] If $\rho_\mu$ is the resistivity in microhm centimeters, then a convenient way of expressing the relaxation time implied by (1.7) is

$$\tau = \left(\frac{0.22}{\rho_\mu}\right)\left(\frac{r_s}{a_0}\right)^3 \times 10^{-14}\text{ sec.} \tag{1.8}$$

Relaxation times calculated from (1.8) and the resistivities in Table 1.2 are displayed in Table 1.3. Note that at room temperatures $\tau$ is typically $10^{-14}$ to $10^{-15}$ sec. In considering whether this is a reasonable number, it is more instructive to contemplate the mean free path, $\ell = v_0\tau$, where $v_0$ is the average electronic speed. The length $\ell$ measures the average distance an electron travels between collisions. In Drude's time it was natural to estimate $v_0$ from classical equipartition of energy: $\frac{1}{2}mv_0{}^2 = \frac{3}{2}k_BT$. Using the known electronic mass, we find a $v_0$ of order $10^7$ cm/sec at room temperature, and hence a mean free path of 1 to 10 Å. Since this distance is comparable to the interatomic spacing, the result is quite consistent with Drude's original view that collisions are due to the electron bumping into the large heavy ions.

However, we shall see in Chapter 2 that this classical estimate of $v_0$ is an order of magnitude too small at room temperatures. Furthermore, at the lowest temperatures in Table 1.3, $\tau$ is an order of magnitude larger than at room temperature, while (as we shall see in Chapter 2) $v_0$ is actually temperature-independent. This can raise the low-temperature mean free path to $10^3$ or more angstroms, about a thousand times the spacing between ions. Today, by working at sufficiently low temperatures with carefully prepared samples, mean free paths of the order of centimeters (i.e., $10^8$ interatomic spacings) can be achieved. This is strong evidence that the electrons do not simply bump off the ions, as Drude supposed.

Fortunately, however, we may continue to calculate with the Drude model without any precise understanding of the cause of collisions. In the absence of a theory of the collision time it becomes important to find predictions of the Drude model that are independent of the value of the relaxation time $\tau$. As it happens, there are several such $\tau$-independent quantities, which even today remain of fundamental interest, for in many respects the precise quantitative treatment of the relaxation time remains the weakest link in modern treatments of metallic conductivity. As a result, $\tau$-independent quantities are highly valued, for they often yield considerably more reliable information.

Two cases of particular interest are the calculation of the electrical conductivity when a spatially uniform static magnetic field is present, and when the electric field

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[11] To convert resistivities from microhm centimeters to statohm centimeters note that a resistivity of 1 $\mu$ohm-cm yields an electric field of $10^{-6}$ volt/cm in the presence of a current of 1 amp/cm$^2$. Since 1 amp is $3 \times 10^9$ esu/sec, and 1 volt is $\frac{1}{300}$ statvolt, a resistivity of 1 $\mu$ohm-cm yields a field of 1 statvolt/cm when the current density is $300 \times 10^6 \times 3 \times 10^9$ esu-cm$^{-2}$-sec$^{-1}$. The statohm-centimeter is the electrostatic unit of resistivity, and therefore gives 1 statvolt/cm with a current density of only 1 esu-cm$^{-2}$-sec$^{-1}$. Thus 1 $\mu$ohm-cm is equivalent to $\frac{1}{9} \times 10^{-17}$ statohm-cm. To avoid using the statohm-centimeter, one may evaluate (1.7) taking $\rho$ in ohm meters, $m$ in kilograms, $n$ in electrons per cubic meter, and $e$ in coulombs. (*Note:* The most important formulas, constants, and conversion factors from Chapters 1 and 2 are summarized in Appendix A.)

Table 1.3
DRUDE RELAXATION TIMES IN UNITS OF $10^{-14}$ SECOND[a]

| ELEMENT | 77 K | 273 K | 373 K |
|---|---|---|---|
| Li | 7.3 | 0.88 | 0.61 |
| Na | 17 | 3.2 | |
| K | 18 | 4.1 | |
| Rb | 14 | 2.8 | |
| Cs | 8.6 | 2.1 | |
| Cu | 21 | 2.7 | 1.9 |
| Ag | 20 | 4.0 | 2.8 |
| Au | 12 | 3.0 | 2.1 |
| Be | | 0.51 | 0.27 |
| Mg | 6.7 | 1.1 | 0.74 |
| Ca | | 2.2 | 1.5 |
| Sr | 1.4 | 0.44 | |
| Ba | 0.66 | 0.19 | |
| Nb | 2.1 | 0.42 | 0.33 |
| Fe | 3.2 | 0.24 | 0.14 |
| Zn | 2.4 | 0.49 | 0.34 |
| Cd | 2.4 | 0.56 | |
| Hg | 0.71 | | |
| Al | 6.5 | 0.80 | 0.55 |
| Ga | 0.84 | 0.17 | |
| In | 1.7 | 0.38 | 0.25 |
| Tl | 0.91 | 0.22 | 0.15 |
| Sn | 1.1 | 0.23 | 0.15 |
| Pb | 0.57 | 0.14 | 0.099 |
| Bi | 0.072 | 0.023 | 0.016 |
| Sb | 0.27 | 0.055 | 0.036 |

[a] Relaxation times are calculated from the data in Tables 1.1 and 1.2, and Eq. (1.8). The slight temperature dependence of $n$ is ignored.

is spatially uniform but time-dependent. Both of these cases are most simply dealt with by the following observation:

At any time $t$ the average electronic velocity $\mathbf{v}$ is just $\mathbf{p}(t)/m$, where $\mathbf{p}$ is the total momentum per electron. Hence the current density is

$$\mathbf{j} = -\frac{ne\mathbf{p}(t)}{m}. \tag{1.9}$$

Given that the momentum per electron is $\mathbf{p}(t)$ at time $t$, let us calculate the momentum per electron $\mathbf{p}(t + dt)$ an infinitesimal time $dt$ later. An electron taken at random at time $t$ will have a collision before time $t + dt$, with probability $dt/\tau$, and will therefore survive to time $t + dt$ without suffering a collision with probability $1 - dt/\tau$. If it experiences no collision, however, it simply evolves under the influence of the force $\mathbf{f}(t)$ (due to the spatially uniform electric and/or magnetic fields) and will therefore acquire an additional momentum[12] $\mathbf{f}(t)\,dt - O(dt)^2$. The contribution of all those electrons that do not collide between $t$ and $t + dt$ to the momentum per electron at time $t + dt$ is the fraction $(1 - dt/\tau)$ they constitute of all electrons, times *their* average momentum per electron, $\mathbf{p}(t) + \mathbf{f}(t)\,dt + O(dt)^2$.

Thus neglecting for the moment the contribution to $\mathbf{p}(t + dt)$ from those electrons that *do* undergo a collision in the time between $t$ and $t + dt$, we have[13]

$$\begin{aligned}\mathbf{p}(t + dt) &= \left(1 - \frac{dt}{\tau}\right)\left[\mathbf{p}(t) + \mathbf{f}(t)dt + O(dt)^2\right] \\ &= \mathbf{p}(t) - \left(\frac{dt}{\tau}\right)\mathbf{p}(t) + \mathbf{f}(t)dt + O(dt)^2.\end{aligned} \tag{1.10}$$

The correction to (1.10) due to those electrons that have had a collision in the interval $t$ to $t + dt$ is only of the order of $(dt)^2$. To see this, first note that such electrons constitute a fraction $dt/\tau$ of the total number of electrons. Furthermore, since the electronic velocity (and momentum) is randomly directed immediately after a collision, each such electron will contribute to the average momentum $\mathbf{p}(t + dt)$ only to the extent that it has acquired momentum from the force $\mathbf{f}$ since its last collision. Such momentum is acquired over a time no longer than $dt$, and is therefore of order $\mathbf{f}(t)dt$. Thus the correction to (1.10) is of order $(dt/\tau)\mathbf{f}(t)dt$, and does not affect the terms of linear order in $dt$. We may therefore write:

$$\mathbf{p}(t + dt) - \mathbf{p}(t) = -\left(\frac{dt}{\tau}\right)\mathbf{p}(t) + \mathbf{f}(t)\,dt + O(dt)^2, \tag{1.11}$$

where the contribution of *all* electrons to $\mathbf{p}(t + dt)$ is accounted for. Dividing this by $dt$ and taking the limit as $dt \to 0$, we find

$$\frac{d\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t). \tag{1.12}$$

This simply states that the effect of individual electron collisions is to introduce a frictional damping term into the equation of motion for the momentum per electron.

We now apply (1.12) to several cases of interest.

## HALL EFFECT AND MAGNETORESISTANCE

In 1879 E. H. Hall tried to determine whether the force experienced by a current carrying wire in a magnetic field was exerted on the whole wire or only upon (what we would now call) the moving electrons in the wire. He suspected it was the latter, and his experiment was based on the argument that "if the current of electricity in a fixed conductor is itself attracted by a magnet, the current should be drawn to one side of the wire, and therefore the resistance experienced should be increased."[14] His

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[12] By $O(dt)^2$ we mean a term of the order of $(dt)^2$.

[13] If the force on the electrons is not the same for every electron, (1.10) will remain valid provided that we interpret $\mathbf{f}$ as the *average* force per electron.

[14] *Am. J. Math.* **2,** 287 (1879).