acquire an additional momentum[12] $\mathbf{f}(t)\,dt - O(dt)^2$. The contribution of all those electrons that do not collide between t and $t + dt$ to the momentum per electron at time $t + dt$ is the fraction $(1 - dt/\tau)$ they constitute of all electrons, times *their* average momentum per electron, $\mathbf{p}(t) + \mathbf{f}(t)\,dt + O(dt)^2$.

Thus neglecting for the moment the contribution to $\mathbf{p}(t + dt)$ from those electrons that *do* undergo a collision in the time between t and $t + dt$, we have[13]

$$\mathbf{p}(t + dt) = \left(1 - \frac{dt}{\tau}\right)\left[\mathbf{p}(t) + \mathbf{f}(t)dt + O(dt)^2\right]$$
$$= \mathbf{p}(t) - \left(\frac{dt}{\tau}\right)\mathbf{p}(t) + \mathbf{f}(t)dt + O(dt)^2. \tag{1.10}$$

The correction to (1.10) due to those electrons that have had a collision in the interval t to $t + dt$ is only of the order of $(dt)^2$. To see this, first note that such electrons constitute a fraction dt/τ of the total number of electrons. Furthermore, since the electronic velocity (and momentum) is randomly directed immediately after a collision, each such electron will contribute to the average momentum $\mathbf{p}(t + dt)$ only to the extent that it has acquired momentum from the force $\mathbf{f}$ since its last collision. Such momentum is acquired over a time no longer than dt, and is therefore of order $\mathbf{f}(t)dt$. Thus the correction to (1.10) is of order $(dt/\tau)\mathbf{f}(t)dt$, and does not affect the terms of linear order in dt. We may therefore write:

$$\mathbf{p}(t + dt) - \mathbf{p}(t) = -\left(\frac{dt}{\tau}\right)\mathbf{p}(t) + \mathbf{f}(t)\,dt + O(dt)^2. \tag{1.11}$$

where the contribution of *all* electrons to $\mathbf{p}(t + dt)$ is accounted for. Dividing this by dt and taking the limit as $dt \to 0$, we find

$$\frac{d\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t). \tag{1.12}$$

This simply states that the effect of individual electron collisions is to introduce a frictional damping term into the equation of motion for the momentum per electron.

We now apply (1.12) to several cases of interest.

HALL EFFECT AND MAGNETORESISTANCE

In 1879 E. H. Hall tried to determine whether the force experienced by a current carrying wire in a magnetic field was exerted on the whole wire or only upon (what we would now call) the moving electrons in the wire. He suspected it was the latter, and his experiment was based on the argument that "if the current of electricity in a fixed conductor is itself attracted by a magnet, the current should be drawn to one side of the wire, and therefore the resistance experienced should be increased."[14] His

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[12] By $O(dt)^2$ we mean a term of the order of $(dt)^2$.

[13] If the force on the electrons is not the same for every electron, (1.10) will remain valid provided that we interpret $\mathbf{f}$ as the *average* force per electron.

[14] *Am. J. Math.* **2**, 287 (1879).

efforts to detect this extra resistance were unsuccessful,[15] but Hall did not regard this as conclusive: "The magnet may *tend* to deflect the current without being able to do so. It is evident that in this case there would exist a state of stress in the conductor, the electricity pressing, as it were, toward one side of the wire." This state of stress should appear as a transverse voltage (known today as the Hall voltage), which Hall was able to observe.

Hall's experiment is depicted in Figure 1.3. An electric field E_x is applied to a wire extending in the x-direction and a current density j_x flows in the wire. In addition, a magnetic field $\mathbf{H}$ points in the positive z-direction. As a result the Lorentz force[16]

$$-\frac{e}{c}\mathbf{v} \times \mathbf{H} \tag{1.13}$$

acts to deflect electrons in the negative y-direction (an electron's drift velocity is *opposite* to the current flow). However the electrons cannot move very far in the y-direction before running up against the sides of the wire. As they accumulate there, an electric field builds up in the y-direction that opposes their motion and their further accumulation. In equilibrium this transverse field (or Hall field) E_y will balance the Lorentz force, and current will flow only in the x-direction.

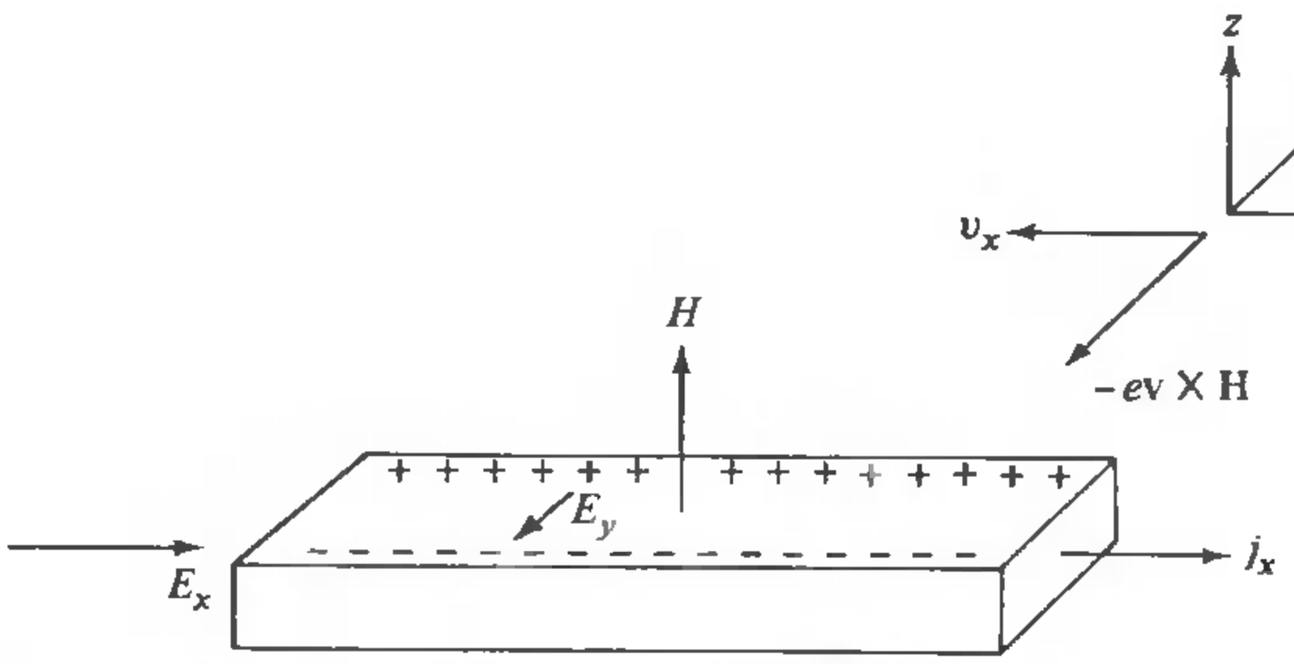

Figure 1.3
Schematic view of Hall's experiment.

There are two quantities of interest. One is the ratio of the field along the wire E_x to the current density j_x,

$$\rho(H) = \frac{E_x}{j_x}. \tag{1.14}$$

This is the magnetoresistance,[17] which Hall found to be field-independent. The other is the size of the transverse field E_y. Since it balances the Lorentz force, one might expect it to be proportional both to the applied field H and to the current along the

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[15] The increase in resistance (known as the magnetoresistance) does occur, as we shall see in Chapters 12 and 13. The Drude model, however, predicts Hall's null result.

[16] When dealing with nonmagnetic (or weakly magnetic) materials, we shall always call the field H, the difference between B and H being extremely small.

[17] More precisely, it is the transverse magnetoresistance. There is also a longitudinal magnetoresistance, measured with the magnetic field parallel to the current.

wire j_x. One therefore defines a quantity known as the Hall coefficient by

$$R_H = \frac{E_y}{j_x H}. \tag{1.15}$$

Note that since the Hall field is in the negative y-direction (Figure 1.3), R_H should be negative. If, on the other hand, the charge carriers were positive, then the sign of their x-velocity would be reversed, and the Lorentz force would therefore be unchanged. As a consequence the Hall field would be opposite to the direction it has for negatively charged carriers. This is of great importance, for it means that a measurement of the Hall field determines the sign of the charge carriers. Hall's original data agreed with the sign of the electronic charge later determined by Thomson. One of the remarkable aspects of the Hall effect, however, is that in some metals the Hall coefficient is positive, suggesting that the carriers have a charge opposite to that of the electron. This is another mystery whose solution had to await the full quantum theory of solids. In this chapter we shall consider only the simple Drude model analysis, which though incapable of accounting for positive Hall coefficients, is often in fairly good agreement with experiment.

To calculate the Hall coefficient and magnetoresistance we first find the current densities j_x and j_y in the presence of an electric field with arbitrary components E_x and E_y, and in the presence of a magnetic field $\mathbf{H}$ along the z-axis. The (position independent) force acting on each electron is $\mathbf{f} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{H}/c)$, and therefore Eq. (1.12) for the momentum per electron becomes[18]

$$\frac{d\mathbf{p}}{dt} = -e\left(\mathbf{E} + \frac{\mathbf{p}}{mc} \times \mathbf{H}\right) - \frac{\mathbf{p}}{\tau}. \tag{1.16}$$

In the steady state the current is independent of time, and therefore p_x and p_y will satisfy

$$0 = -eE_x - \omega_c p_y - \frac{p_x}{\tau}, \tag{1.17}$$

$$0 = -eE_y + \omega_c p_x - \frac{p_y}{\tau},$$

where

$$\omega_c = \frac{eH}{mc}. \tag{1.18}$$

We multiply these equations by $-ne\tau/m$ and introduce the current density components through (1.4) to find

$$\begin{aligned} \sigma_0 E_x &= \omega_c \tau j_y + j_x, \\ \sigma_0 E_y &= -\omega_c \tau j_x + j_y, \end{aligned} \tag{1.19}$$

where σ_0 is just the Drude model DC conductivity in the absence of a magnetic field, given by (1.6).

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[18] Note that the Lorentz force is not the same for each electron since it depends on the electronic velocity $\mathbf{v}$. Therefore the force $\mathbf{f}$ in (1.12) is to be taken as the average force per electron (see Footnote 13). Because, however, the force depends on the electron on which it acts only through a term *linear* in the electron's velocity, the average force is obtained simply by replacing that velocity by the average velocity, $\mathbf{p}/m$.

The Hall field E_y is determined by the requirement that there be no transverse current j_y. Setting j_y to zero in the second equation of (1.19) we find that

$$E_y = -\left(\frac{\omega_c\tau}{\sigma_0}\right)j_x = -\left(\frac{H}{nec}\right)j_x. \tag{1.20}$$

Therefore the Hall coefficient (1.15) is

$$R_H = -\frac{1}{nec}. \tag{1.21}$$

This is a very striking result, for it asserts that the Hall coefficient depends on no parameters of the metal except the density of carriers. Since we have already calculated n assuming that the atomic valence electrons become the metallic conduction electrons, a measurement of the Hall constant provides a direct test of the validity of this assumption.

In trying to extract the electron density n from measured Hall coefficients one is faced with the problem that, contrary to the prediction of (1.21), they generally do depend on magnetic field. Furthermore, they depend on temperature and on the care with which the sample has been prepared. This result is somewhat unexpected, since the relaxation time τ, which can depend strongly on temperature and the condition of the sample, does not appear in (1.21). However, at very low temperatures in very pure, carefully prepared samples at very high fields, the measured Hall constants do appear to approach a limiting value. The more elaborate theory of Chapters 12 and 13 predicts that for many (but not all) metals this limiting value is precisely the simple Drude result (1.21).

Some Hall coefficients at high and moderate fields are listed in Table 1.4. Note the occurrence of cases in which R_H is actually positive, apparently corresponding to carriers with a positive charge. A striking example of observed field dependence totally unexplained by Drude theory is shown in Figure 1.4.

The Drude result confirms Hall's observation that the resistance does not depend on field, for when $j_y = 0$ (as is the case in the steady state when the Hall field has been established), the first equation of (1.19) reduces to $j_x = \sigma_0 E_x$, the expected result for the conductivity in zero magnetic field. However, more careful experiments on a variety of metals have revealed that there is a magnetic field dependence to the resistance, which can be quite dramatic in some cases. Here again the quantum theory of solids is needed to explain why the Drude result applies in some metals and to account for some truly extraordinary deviations from it in others.

Before leaving the subject of DC phenomena in a uniform magnetic field, we note for future applications that the quantity $\omega_c\tau$ is an important, dimensionless measure of the strength of a magnetic field. When $\omega_c\tau$ is small, Eq. (1.19) gives $\mathbf{j}$ very nearly parallel to $\mathbf{E}$, as in the absence of a magnetic field. In general, however, $\mathbf{j}$ is at an angle ϕ (known as the Hall angle) to $\mathbf{E}$, where (1.19) gives $\tan\phi = \omega_c\tau$. The quantity ω_c, known as the cyclotron frequency, is simply the angular frequency of revolution[19]

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[19] In a uniform magnetic field the orbit of an electron is a spiral along the field whose projection in a plane perpendicular to the field is a circle. The angular frequency ω_c is determined by the condition that the centripetal acceleration $\omega_c^2 r$ be provided by the Lorentz force, $(e/c)(\omega_c r)H$.

Table 1.4
HALL COEFFICIENTS OF SELECTED ELEMENTS IN MODERATE TO HIGH FIELDS[a]

METAL	VALENCE	$-1/R_H nec$
Li	1	0.8
Na	1	1.2
K	1	1.1
Rb	1	1.0
Cs	1	0.9
Cu	1	1.5
Ag	1	1.3
Au	1	1.5
Be	2	−0.2
Mg	2	−0.4
In	3	−0.3
Al	3	−0.3

[a] These are roughly the limiting values assumed by R_H as the field becomes very large (of order 10^4 G), and the temperature very low, in carefully prepared specimens. The data are quoted in the form n_0/n, where n_0 is the density for which the Drude form (1.21) agrees with the measured R_H: $n_0 = -1/R_H ec$. Evidently the alkali metals obey the Drude result reasonably well, the noble metals (Cu, Ag, Au) less well, and the remaining entries, not at all.

of a free electron in the magnetic field H. Thus $\omega_c\tau$ will be small if electrons can complete only a small part of a revolution between collisions, and large if they can complete many revolutions. Alternatively, when $\omega_c\tau$ is small the magnetic field deforms the electronic orbits only slightly, but when $\omega_c\tau$ is comparable to unity or larger, the effect of the magnetic field on the electronic orbits is quite drastic. A useful numerical evaluation of the cyclotron frequency is

$$\nu_c\,(10^9 \text{ hertz}) = 2.80 \times H \text{ (kilogauss)}, \qquad \omega_c = 2\pi\nu_c. \tag{1.22}$$

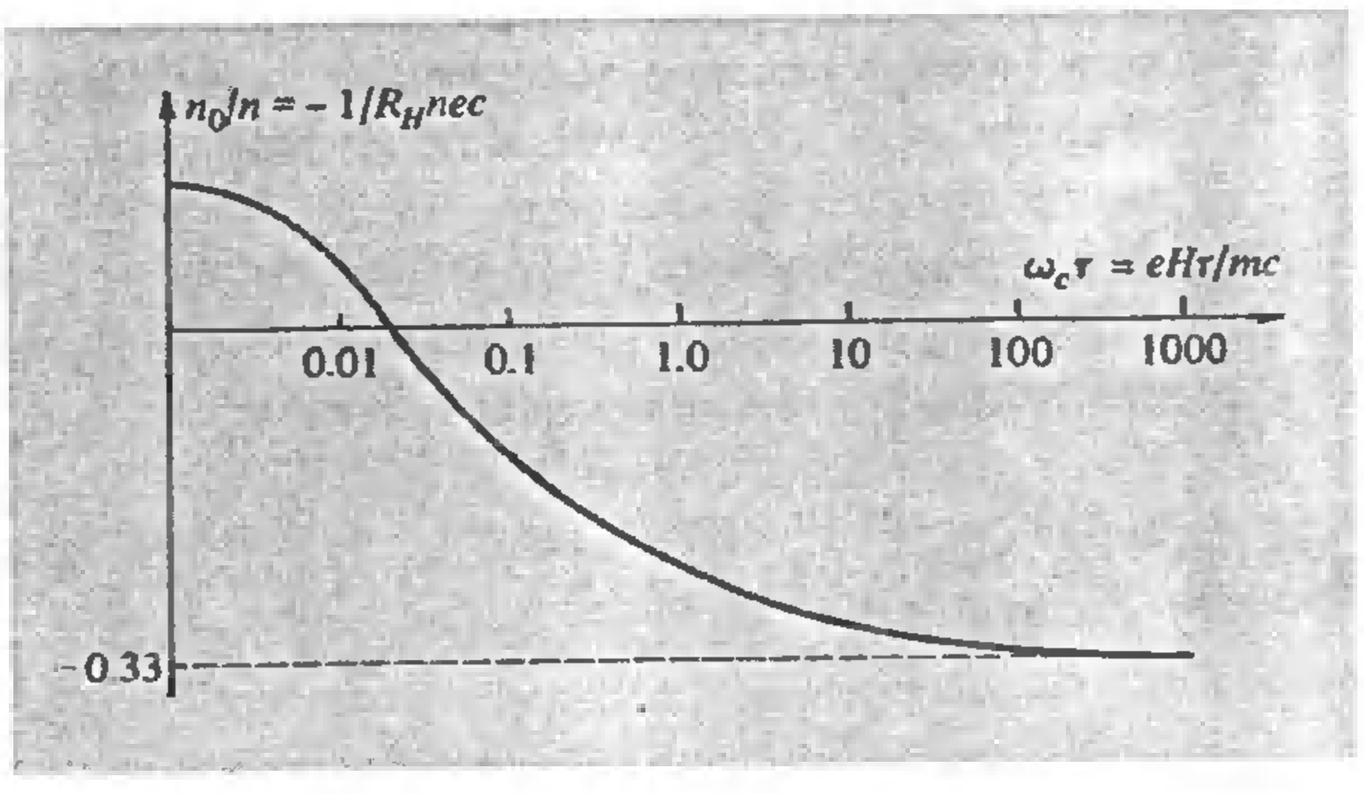

Figure 1.4
The quantity $n_0/n = -1/R_H nec$, for aluminum, as a function of $\omega_c\tau$. The free electron density n is based on a nominal chemical valence of 3. The high field value suggests only one carrier per primitive cell, with a positive charge. (From R. Lück, *Phys. Stat. Sol.* **18**, 49 (1966).)