
1 Survey of basic principles

Although magnetoresistance is not intrinsically a low temperature phenomenon like superconductivity or superfluidity, in practice with magnetic field strengths available in the laboratory it is hardly likely to attract attention except in rather pure materials at low temperatures. This is not so true in semiconductors as in metals, which are our primary concern, or in the semi-metal bismuth whose sensitivity to modest magnetic fields, even at room temperature, has made it a useful compact field-measuring instrument since before 1886.⁽¹⁾ The effect is not dramatic, about 18% increase in resistivity in a transverse field of 0.6 T, rising to a 40-fold change at 24 T as observed by Kapitza.⁽²⁾ Copper is more typical in that the same very powerful field gave rise to a change of only 2% at room temperature.⁽³⁾ Cooling to the temperature (4 K) of liquid helium works wonders – a reasonably pure sample of polycrystalline copper was found to increase its resistance 14-fold in a field of 10 T,⁽⁴⁾ and the better material available nowadays might be expected to show a change at least 5 times larger.⁽⁵⁾ As for bismuth, a pure sample can have its resistance changed by a factor of several million by applying a field of 10 T.⁽⁶⁾

To estimate the conditions necessary for a marked magnetoresistance, consider a condensed Fermi gas of electrons with an approximately spherical Fermi surface. The conductivity and resistivity are given by the well-known formulae

$$\sigma_0 = ne^2\tau/m^* \quad \text{and} \quad \rho_0 = m^*/ne^2\tau, \quad (1.1)$$

in which n is the number of electrons per unit volume and m^* their effective mass, not necessarily the real electron mass. The relaxation time τ describes the time-constant for a current to die away when the sustaining electric field is removed. When $B=0$ the electrons travel in straight lines between collisions, and a magnetic field can only have a significant effect on the conductivity if it is strong enough to bend the trajectory appreciably during a free path. The Lorentz force $e\mathbf{v} \wedge \mathbf{B}$ bends the paths into helices whose

axes are parallel to \mathbf{B} , and the angular velocity of an electron round its particular axis is the cyclotron frequency

$$\omega_c = eB/m^* \quad (1.2)$$

The mean angle turned between collisions is $\omega_c \tau$, and unless $\omega_c \tau > 1$ no great magnetoresistance effect can be expected. As we shall see presently, a large value of $\omega_c \tau$ does not guarantee magnetoresistance, but generally speaking the criterion is sound. Combining (1) and (2) we have

$$\omega_c \tau = B\sigma_0/ne, \quad (1.3)$$

from which the least well known quantity, m^* , has disappeared, allowing an estimate of $\omega_c \tau$. In copper, for example, with one conduction electron per atom, the atomic volume is $7.8 \times 10^{-3} \text{ m}^3/\text{kg mole}$, so that $n = 8.5 \times 10^{28} \text{ m}^{-3}$. At 0°C , $\sigma_0 = 6.4 \times 10^7 \Omega^{-1} \text{ m}^{-1}$ and hence, from (3), $\omega_c \tau = 4.7 \times 10^{-3} \text{ B}$. In the strongest field attained by Kapitza, $B = 30 \text{ T}$, $\omega_c \tau$ was only 0.14 and it is hardly surprising the observed effect was so small. The sample used by de Launay *et al.*,⁽⁴⁾ however, at 4 K had a conductivity 606 times higher so that, when $B = 10 \text{ T}$, $\omega_c \tau$ was 28; the electrons are now wound into tight enough helices to execute about 4 turns between collisions, easily enough to change the conduction process considerably.

The strong effect in bismuth at room temperature arises because, containing as it does only about 10^{-5} conduction electrons per atom,⁽⁷⁾ its resistivity is nevertheless no more than 70 times that of copper – an indication of an unusually long relaxation time. If we overlook the fact that the Fermi surface is far from spherical, so that the use of (3) can give at best a rough estimate, the outcome of its application here is that $\omega_c \tau \sim 20 \text{ B}$, and the occurrence of magnetoresistance, even at room temperature, is made plausible. In this case conditions are especially favourable, in that bismuth is a compensated metal, with equal numbers of electrons and holes, and such metals show the strongest magnetoresistance of all. At the other extreme lie those metals which have electrons only, and no holes, and where the Fermi surface is nearly spherical. Potassium, carefully handled, closely approximates to this ideal free-electron model and has only a very weak magnetoresistance. The ideal should show none at all, a paradoxical result that will be explained in the next section.

Direct measurement of the cyclotron frequency, combined with (2), enables one to derive an effective mass (or *cyclotron mass*, m_c) to be compared with the mass m_e of a free electron. In general a metal is not characterized by a single value of m_c/m_e , different orbits presenting different values. In bismuth, when \mathbf{B} lies along the bisectrix, it can be as low as 8×10^{-3} , but there are other orbits for which it is 0.21.⁽⁷⁾ In the noble and

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alkali metals it is not far from unity for the majority of orbits; but in transition and rare earth metals the electrons in the lower bands frequently have m_c/m_e considerably in excess of unity. And the mass is still further enhanced if the electron carries a clothing of virtual phonons with it, being as high as 90 in U Pt₃.⁽⁸⁾

The free-electron gas shows no magnetoresistance

In the free-electron model the electrons are supposed to move independently, obeying Newton's laws of motion and possessing a mass m (or m^* in the quasi-free model) which is isotropic. Extension to anisotropic mass, with an acceleration law $m_{ij} \dot{v}_j = F_i$, is easy, but we shall not concern ourselves at the moment with this rather artificial concept. The argument that follows does not depend greatly on the electrons forming a degenerate Fermi gas as in most metals at ordinary or low temperatures – it is equally good for the rarefied Boltzmann gas typical of pure semiconductors. Whatever the distribution of velocities, each electron is accelerated by an electric field according to the equation

$$m^* \dot{\mathbf{v}} = e \mathcal{E}. \quad (1.4)$$

If the electrons are not deflected, and their motion randomized, by collisions the current density rises steadily,

$$\dot{\mathbf{J}} = \sum e \dot{\mathbf{v}} = ne^2 \mathcal{E} / m^*, \quad (1.5)$$

the summation being taken over all electrons in unit volume.

Now let us introduce collisions by supposing that \mathbf{J} , after \mathcal{E} is removed, would decay exponentially. This is a good approximation for an isotropic degenerate gas, and fairly good for a Boltzmann gas, and it leads us to extend (5) to read

$$\dot{\mathbf{J}} = ne^2 \mathcal{E} / m^* - \mathbf{J} / \tau, \quad (1.6)$$

τ being assumed to take the same value for all directions of \mathbf{J} . In the steady state $\dot{\mathbf{J}} = 0$, and $\mathbf{J} = \sigma_0 \mathcal{E}$, with σ_0 having the same form as in (1).

It is helpful for the next stage of the argument to consider the momentum balance in the electron gas. The momentum density $\mathbf{P} = \sum m^* \mathbf{v} = m^* \mathbf{J} / e$, and (6) may be rewritten

$$\dot{\mathbf{P}} = ne \mathcal{E} - \mathbf{P} / \tau. \quad (1.7)$$

The momentum density is subject to change from two force-densities, the electrical force-density $ne \mathcal{E}$ and the collisional $-\mathbf{P} / \tau$ which is the Newtonian reaction of the collision centres in the lattice when struck by

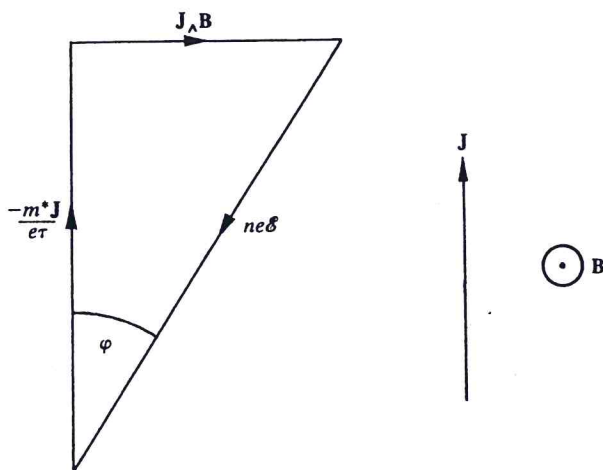


Figure 1.1 Forces acting on unit volume of a quasi-free electron gas carrying current J in transverse field B .

electrons. In the steady state, of course, they balance.[†] Now let B be applied transverse to J , exerting a Lorentz force $e\mathbf{v} \wedge \mathbf{B}$ on each electron and a force-density $\sum e\mathbf{v} \wedge \mathbf{B}$, i.e. $\mathbf{J} \wedge \mathbf{B}$, on the electron assembly as a whole. Adding this into (7) we see that in the steady state,

$$0 = ne\mathcal{E} - \mathbf{P}/\tau + \mathbf{J} \wedge \mathbf{B} = ne\mathcal{E} - m^*\mathbf{J}/e\tau + \mathbf{J} \wedge \mathbf{B}. \quad (1.8)$$

In representing this result as a vector diagram (fig. 1) it has been assumed that neither m^* nor τ are affected by B – almost always a safe assumption. One sees immediately that the component of \mathcal{E} parallel to J is unchanged, but a transverse component arises to cancel the Lorentz force. This is the Hall⁽⁹⁾ field, of magnitude JB/ne . In an experiment where a current is passed along a wire or strip the direction of J is predetermined, and \mathcal{E} adjusts itself, with the aid of space charges or surface charges if necessary, to satisfy (8). Then potential contacts can be placed to respond either to the parallel component, giving the resistance (unaffected by B in this case) or the Hall field. The angle φ between J and \mathcal{E} , the Hall angle, may also be measured in this way. The diagram shows that

$$\tan \varphi = eB\tau/m^* = \omega_c\tau \quad \text{from (2)}. \quad (1.9)$$

Alternatively, and especially in semiconductor physics, $e\tau/m^*$ may be

205

[†] The correctness of this treatment depends on the convection current of momentum being non-divergent. If this is not the case, as in an example treated in chapter 6, an extra term must be added to the momentum balance. Here, however, spatial uniformity makes it unnecessary.

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$$\rho_{xx} = \sigma_{yy}/\Delta'(\epsilon)$$

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written as μ , the mobility, being the drift velocity acquired by an electron in an electric field of unit strength; and $\tan \varphi = \mu B$.

The magnetoconductivity and magnetoresistivity tensors

When the direction and magnitude of \mathbf{J} are fixed by the experimental procedure, and \mathcal{E} adjusts itself accordingly, what is measured is the resistivity tensor, $\rho_{ij}J_j$, or a component or combination of components;

$$\mathcal{E}_i = \rho_{ij}J_j. \quad (1.10)$$

For theoretical purposes, however, it is usually more convenient to imagine \mathcal{E} applied in some chosen direction and to calculate the resulting \mathbf{J} , related to it by the conductivity tensor,

$$J_i = \sigma_{ij}\mathcal{E}_j. \quad (1.11)$$

The relation between σ_{ij} and its inverse ρ_{ij} is

$$\rho_{ii} = (\sigma_{jj}\sigma_{kk} - \sigma_{jk}\sigma_{kj})/\Delta(\sigma) \quad ; \quad \rho_{ij} = (\sigma_{ik}\sigma_{kj} - \sigma_{ij}\sigma_{kk})/\Delta(\sigma) \quad (1.12)$$

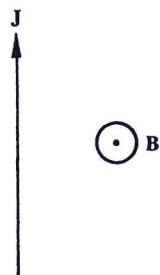
in which $\Delta(\sigma)$ is the determinant whose elements are the elements of σ_{ij} . Similarly,

$$\sigma_{ii} = (\rho_{jj}\rho_{kk} - \rho_{jk}\rho_{kj})/\Delta(\rho) \quad ; \quad \sigma_{ij} = (\rho_{ik}\rho_{kj} - \rho_{ij}\rho_{kk})/\Delta(\rho). \quad (1.13)$$

In the absence of a magnetic field, σ_{ij} and ρ_{ij} are symmetrical, i.e. $\sigma_{ij} = \sigma_{ji}$, and this implies that orthogonal axes can be found, with reference to which they are diagonal. In cubic, tetragonal or orthorhombic crystals the edges of the unit cell are automatically axes with this property. In cubic crystals $\sigma_{ii} = \sigma_{jj} = \sigma_{kk}$, and both ρ_{ij} and σ_{ij} are isotropic – for all directions of \mathcal{E} , $\mathbf{J} = \sigma\mathcal{E}$ and is parallel to \mathcal{E} . In tetragonal crystals two diagonal elements are the same, and in orthorhombic crystals all are different. Then with reference to different axes the off-diagonal elements do not in general vanish. Hexagonal crystals behave like tetragonal in that σ_{zz} , along the hexad axis, is different from the other two which are identical.

When B is present σ_{ij} and ρ_{ij} are not in general symmetrical, and require all nine elements of each for a complete specification. Quite frequently, however, the coupling between longitudinal and transverse effects is small enough to be neglected; that is to say, \mathcal{E}_z applied parallel to \mathbf{B} produces only J_z with negligible J_x and J_y , while if \mathcal{E} lies transverse, in the plane normal to \mathbf{B} , so also does \mathbf{J} , though not necessarily parallel to \mathcal{E} . This verbal description is the same as putting $\sigma_{xz}, \sigma_{zx}, \sigma_{yz}, \sigma_{zy}$ all equal to zero; and similarly for the same components of ρ_{ij} . Then (12) may be written

$$\left. \begin{aligned} \rho_{xx} &= \sigma_{yy}/\Delta'(\sigma), & \rho_{yy} &= \sigma_{xx}/\Delta'(\sigma), & \rho_{zz} &= 1/\sigma_{zz}, \\ \rho_{xy} &= -\sigma_{xy}/\Delta'(\sigma), & \rho_{yx} &= -\sigma_{yx}/\Delta'(\sigma), \end{aligned} \right\} \quad (1.14)$$



of a quasi-free electron gas

ence.† Now let \mathbf{B} be applied on each electron and a forceably as a whole. Adding this

$$+\mathbf{J} \wedge \mathbf{B}. \quad (1.8)$$

(fig. 1) it has been assumed always a safe assumption. parallel to J is unchanged, the Lorentz force. This is the point where a current is passed defined, and \mathcal{E} adjusts itself, as if necessary, to satisfy (8). depend either to the parallel to \mathbf{B} in this case) or the Hall effect, may also be measured in

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where

$$\Delta'(\sigma) = \sigma_{xx}\sigma_{yy} - \sigma_{xy}\sigma_{yx};$$

and, of course, similarly for inverting ρ_{ij} into σ_{ij} .

The behaviour represented in fig. 1 may be written, by use of (9),

$$\rho_{xx} = \rho_{yy} = 1/\sigma_0 \quad \text{and} \quad \rho_{xy} = -\rho_{yx} = \omega_c\tau/\sigma_0. \quad (1.15)$$

Also, since the conductivity along the direction of B is unaffected, $\rho_{zz} = 1/\sigma_0$, and there is no longitudinal-transverse coupling. Hence (14) applies here, and

$$\sigma_{xx} = \sigma_{yy} = \sigma_0/(1 + \gamma^2) \quad \text{and} \quad \sigma_{xy} = -\sigma_{yx} = -\gamma\sigma_0/(1 + \gamma^2) \quad (1.16)$$

where $\gamma = \omega_c\tau$. Alternatively, we may use (9) to write

$$\sigma_{xx} = \sigma_{yy} = \sigma_0 \cos^2 \phi \quad \text{and} \quad \sigma_{xy} = -\sigma_{yx} = -\frac{1}{2}\sigma_0 \sin 2\phi. \quad (1.17)$$

When the direction of \mathcal{E} is predetermined, rather than J as in fig. 1, the corresponding diagram is that in fig. 2, the semicircle on which the end of J lies being drawn with diameter $\sigma_0\mathcal{E}$. With increase of B the magnitude of J decreases steadily according to (16), but the resistivity ρ_{xx} remains constant since the component of \mathcal{E} parallel to J decreases at the same rate.

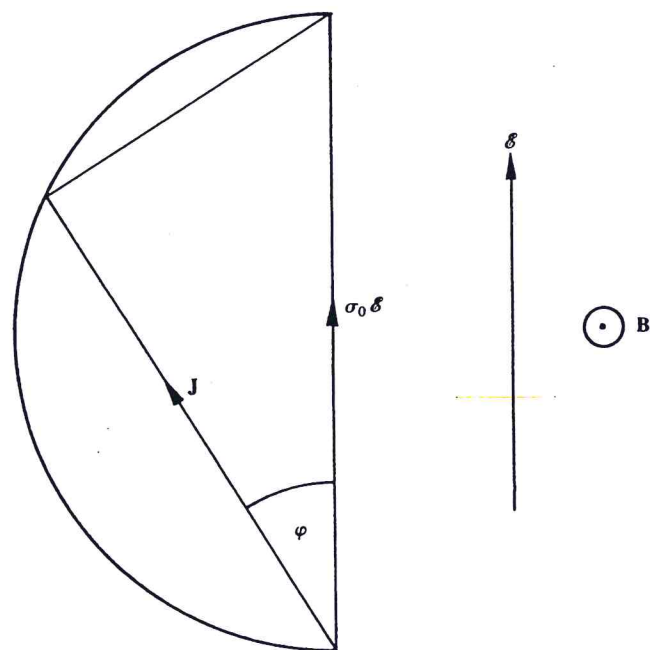


Figure 1.2 Relationship between J and \mathcal{E} for a quasi-free electron gas in transverse field B .

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$$\rho = 1/\sigma = (\sigma_1 + i\sigma_2)/(\sigma_1^2 + \sigma_2^2)$$

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Corresponding to (15) and (16,
 $\gamma > 0$),

$$\rho = \rho_0(1 - i\gamma) \quad \text{and}$$

In (15)–(17) the signs of σ_{xy} , etc. depend on the sign of $\omega_c\tau$. In the diagrams in this book where cartesian coordinates are used or implied, z and \mathbf{B} will point, wherever convenient, out of the page. A free electron moving in the plane of the page describes an anticlockwise orbit. The Hall coefficient R_H , defined as having magnitude $|\rho_{xy}/B_z|$, is conventionally taken to be negative for free electrons and for any conductor in which φ has the same sign as for a free-electron metal. Only rarely shall we be interested in the sign of R_H and no trouble will be taken to maintain a consistent convention – the sign can usually be adjusted by inspection at the end of any calculation.

In real metals, if \mathbf{B} happens to lie along an axis of threefold or fourfold symmetry, σ_{ij} automatically possesses the properties represented in (14), being invariant with respect to rotation of the axes around \mathbf{B} . For rotation through ϕ makes the following transformations:

$$\sigma_{xx} \rightarrow \sigma_{xx} \cos^2 \phi - (\sigma_{xy} + \sigma_{yx}) \cos \phi \sin \phi + \sigma_{yy} \sin^2 \phi$$

and

$$\sigma_{xy} \rightarrow \sigma_{xy} \cos^2 \phi + (\sigma_{xx} - \sigma_{yy}) \cos \phi \sin \phi - \sigma_{yx} \sin^2 \phi.$$

If the axis has threefold symmetry, putting $\phi = 120^\circ$ must leave σ_{xx} and σ_{xy} unchanged, and it is easily seen that this requires $\sigma_{yy} = \sigma_{xx}$ and $\sigma_{xy} = -\sigma_{yx}$; similarly for $\phi = 90^\circ$, if there is fourfold symmetry about \mathbf{B} . Also there is no longitudinal–transverse coupling and σ_{ij} is specified by only three components which we shall frequently write as $\sigma_1 = \sigma_{xx} = \sigma_{yy}$, $\sigma_2 = \sigma_{xy} = -\sigma_{yx}$ and $\sigma_3 = \sigma_{zz}$; that is to say,

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & \sigma_2 & 0 \\ -\sigma_2 & \sigma_1 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \quad (1.18)$$

and correspondingly for ρ_{ij} .

It is sometimes convenient to treat the plane normal to \mathbf{B} as a complex plane in which \mathcal{E} and \mathbf{J} are represented by complex numbers. They are linearly related by complex σ or ρ , $\sigma = \sigma_1 - i\sigma_2$ and $\rho = \rho_1 - i\rho_2$. The negative sign arises because $J_y = \sigma_{yx}\mathcal{E}_x = \text{Im}[\sigma\mathcal{E}]$ when \mathcal{E} lies along the real axis. The inversion of σ into ρ is now a simple matter of complex algebra;

$$\rho = 1/\sigma = (\sigma_1 + i\sigma_2)/(\sigma_1^2 + \sigma_2^2) \quad (1.19)$$

so that $\rho_1 = \sigma_1/(\sigma_1^2 + \sigma_2^2)$ and $\rho_2 = -\sigma_2/(\sigma_1^2 + \sigma_2^2)$ in agreement with (14). Corresponding to (15) and (16) we have, for a free-electron metal (for which $\gamma > 0$),

$$\rho = \rho_0(1 - i\gamma) \quad \text{and} \quad \sigma = \sigma_0/(1 - i\gamma). \quad (1.20)$$