From (37), disregarding signs,

$$\sigma_{xy} = (e^2/4\pi^3\hbar)\int L_x \mathrm{d}S_y = (e/4\pi^3 B)\int k_y \mathrm{d}S_y.$$

Since dS is directed normal to the Fermi surface, $\mathrm{d}S_y$ for a slice δk_z thick can be written as $\delta k_z \mathrm{d}k_x$, and the integral becomes a line integral, $\oint k_y \mathrm{d}k_x$, which is just $\mathscr{A}_k$, the area of the k-orbit. Then, since $\mathscr{A}_k \delta k_z/4\pi^3$ is the number of electrons per unit volume whose k lies in the slice, we may integrate over the whole Fermi surface:

$$\sigma_{xy} = ne/B. \tag{1.50}$$

We shall see later that frequently σ_{xx} and σ_{yy} vary in high fields as $1/B^2$, and then, as (14) shows and the free-electron gas illustrates,

$$\rho_{yx} \sim 1/\sigma_{xy} \sim B/ne, \tag{1.51}$$

so that the limiting form of the Hall constant is

$$R_{\mathrm{H}} \sim 1/ne, \tag{1.52}$$

the same as for a free-electron gas, even though the energy surfaces may be entirely different. All we ask is that they be closed, so that $\mathscr{A}_k$ has a meaning for all sections.

The expression (52) applies at all values of **B** to a free-electron gas, but in general only in the limit of high **B**; in the next section we shall meet an example where it describes the high-field behaviour well but gives even the sign of the Hall field incorrectly when **B** is smaller. There are many ways of deriving (52),[29] some of which make use of the fact that the result is independent of scattering, which may therefore be ignored, if convenient. In fields $\mathscr{E}_x$ and B_z any charged particle has drift velocity $v_y = \mathscr{E}_x/B_z$ superimposed on its orbital motion, and (50) is an expression of this behaviour. But there is no need to go further into alternative approaches to the simplest of the problems in magnetoresistivity.

Elementary extensions of the free-electron theory

1. *Electrons and holes*

The concept of a hole, with properties that simulate a positively charged electron, is too well established in semiconductor physics to need careful exposition at this point. All the same, its use in connection with metals is sufficiently different in some respects from what is usually understood that we must discuss it in more detail at the right time, in chapter 3. It is enough at present to consider a metal in which there are two independent types of particle, both behaving classically, and differing only in charge, i.e. n_- electrons and n_+ holes per unit volume. If each were present alone, it would confer positive conductivity on the metal in zero magnetic field, σ_- for the electrons and σ_+ for the holes; and in the presence of a transverse field **B** each would behave according to (16), or (20) if complex notation is used. Thus for the two together,

$$\sigma = \sigma_-/(1 - \mathrm{i}\gamma_-) + \sigma_+/(1 + \mathrm{i}\gamma_+), \tag{1.53}$$

in which γ is written for $|\omega_c\tau|$ and the opposite charges are reflected in the signs in the denominators. By use of (3) we rewrite this:

$$\sigma = (n_-e/B)[\gamma_-/(1 - \mathrm{i}\gamma_-) + c\gamma_+/(1 + \mathrm{i}\gamma_+)] \tag{1.54}$$

in which $c = n_+/n_-$. If the density of holes equals that of electrons, $c = 1$ and the metal is said to be compensated.

The resistivity ρ is $1/\sigma$ and looks more complicated:

$$\rho = \frac{B}{n_-e}\frac{[\gamma_- + c\gamma_+ + \gamma_-\gamma_+(\gamma_+ + c\gamma_-)] - \mathrm{i}[\gamma_-^2 - c\gamma_+^2 + (1-c)\gamma_-^2\gamma_+^2]}{(\gamma_- + c\gamma_+)^2 + (1-c)^2\gamma_-^2\gamma_+^2}. \tag{1.55}$$

The low and high field limits are easily found by keeping only terms up to $\gamma^2(\propto B^2)$ for the former, and only the highest orders in γ for the latter. Then, writing ρ_0 doe $(\sigma_- + \sigma_+)^{-1}$, and ρ_1 for ρ_{xx}, the real part of ρ, we have

$$(\rho_1/\rho_0)_{\text{low}} \sim 1 + c\gamma_-\gamma_+(\gamma_- + \gamma_+)^2/(\gamma_- + c\gamma_+)^2, \tag{1.56}$$

describing a quadratic magnetoresistance, $\Delta\rho_1/\rho_0 \propto B^2$. Unless $c = 1$ the increase is not continued indefinitely, but eventually saturation occurs at a value ρ_∞:

$$\rho_\infty/\rho_0 \sim A/(1-c)^2, \quad \text{where } A = (\sigma_- + \sigma_+)(1/\sigma_- + c^2/\sigma_+). \tag{1.57}$$

In a compensated metal (55) simplifies, by use of (54) and without approximation, to a pure quadratic effect:

$$(\rho_1/\rho_0)_{\text{comp}} = 1 + \gamma_-\gamma_+. \tag{1.58}$$

When compensation is not quite perfect, so that $c = 1 - \varepsilon$ $(\varepsilon \ll 1)$, the high field limit (57) is of the order of $4/\varepsilon^2$, which may be very large but is not reached until B itself is large. The quadratic rise described by (58) would need to continue until $\gamma \sim 2/\varepsilon$ before it met the saturation value. This can be illustrated by choosing $\gamma_+ = \gamma_- = \gamma$ in (55) when, without approximation,

$$\rho_1/\rho_0 = (1 + \gamma^2)/[1 + \gamma^2\varepsilon^2/(1 + c)^2]. \tag{1.59}$$

Examples are shown in fig. 16. When ε is small the initial stages of the curve

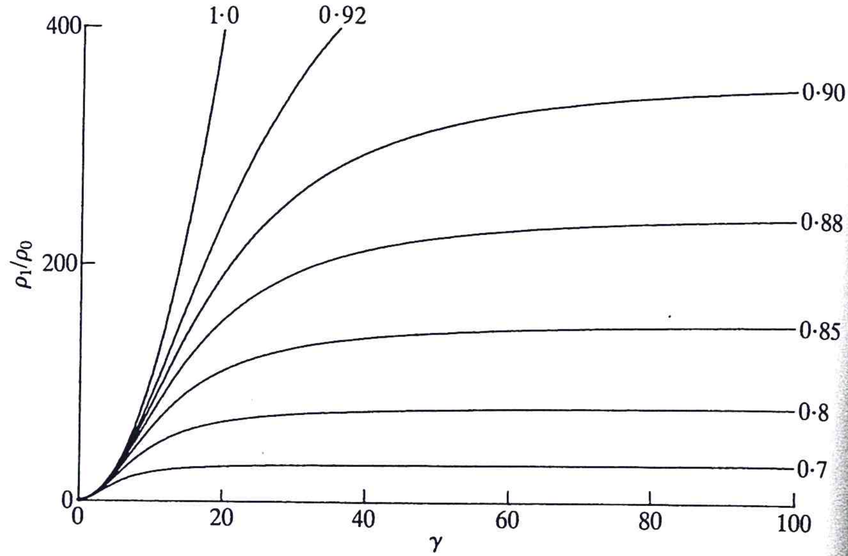

Figure 1.16 Transverse magnetoresistance of a metal containing both electrons and holes, according to (59); values of c, i.e. n_+/n_-, are given beside each curve.

9 differ little, following (58), so that $\Delta\rho_1/\rho_0 \sim (\omega_c\tau)^2$ as has already been noted as an empirical relation. When, as in bismuth,[(6)] the quadratic rise has been found to continue until $\rho_1/\rho_0 > 10^6$, ε must be less than 10^{-3}; with only about 10^{-5} electrons per atom, the material must have been pure enough for the numbers of electrons and holes to differ by, at most, 10^{-8} per atom. As is well known in semiconductor physics, this degree of purity demands great attention to sample purification.

The same is true for the Hall effect, as described by the imaginary part of (55). Unless $c = 1$ the high-field limit is

$$\rho_{yx} \sim B/ne(1-c) = B/e(n_- - n_+) \tag{1.60}$$

which is a generalization of (51). When $\varepsilon \ll 1$ a high field is needed for this limit to be reached; until this is achieved (and always when $c = 1$), terms of lower order in γ must be kept when approximating to (55). If $c = 1$, without approximation,

$$\rho_{yx} = (B/n_-e)(\gamma_- - \gamma_+)/(\gamma_- + \gamma_+). \tag{1.61}$$

In bismuth the electrons are considerably more mobile than the holes, and $\gamma_- \gg \gamma_+$. In a perfectly compensated sample, then, one expects from (61) that the sign of the Hall effect will reflect the dominance of electrons in the conduction process. A not-very-pure sample, however, in which holes

slightly outnumber electrons must ultimately settle down in the form determined by (60) with the Hall field of opposite sign. In the end numbers are more important than mobility. Reversal of the sign of the Hall effect is not uncommon in bismuth samples.[30] It is always found in aluminium,[31] 93
which is not compensated and has more holes than electrons; it is this fact that controls the high-field behaviour. At low fields, however, it behaves more like a free-electron metal with three electrons per atom. The simple model used here is now inadequate, and the same must be said of the curiosities of Hall 'constant' in magnesium, zinc and other metals suffering 137
magnetic breakdown. In later chapters these matters will receive more realistic appraisal.

In an uncompensated metal the Hall angle, φ in fig. 1, rises to $\pi/2$, while 4
in a compensated metal it tends to remain small and eventually falls to zero. This behaviour follows immediately from the high-field expressions (57) and (60) for the uncompensated metal, and (58) and (61) for the compensated. In the former $\tan\varphi \propto B$, in the latter $\tan\varphi \propto 1/B$. The Hall effect may be hard to measure in a compensated metal since the transverse component of $\mathscr{E}$ is commonly much smaller than the longitudinal. The reverse holds in an uncompensated metal, where the Hall field may be large and create difficulties in measuring the longitudinal component and hence the magnetoresistivity. Techniques for overcoming the problems, and the new problems raised by the techniques themselves, are the subject of chapter 2.

2. *Anisotropic scattering*

The simple argument incorporated in fig. 1 depends for its validity on the relaxation time being independent of the direction of **J**. Since currents in different directions involve different displacements of the Fermi surface, variations of scattering rate over the surface are likely to cause anisotropy in the relaxation rate of **J**, unless crystal symmetry so constrains those variations that σ_{ij} must be isotropic. Most cases of anisotropic scattering require long and tedious analysis to work out fully, but a single simplified model will serve to show that the effects to be expected are rather insignificant in comparison with most of the examples of interest, e.g. figs. 5, 6 and 10. Even with the most drastic simplification it is still desirable to concentrate on one feature only, the ratio ρ_∞/ρ_0.

Consider then a two-dimensional free-electron gas for which the probability of an electron being scattered catastrophically in time δt varies with its direction of motion, ϕ, as $(p_0 + p_1 \cos 2\phi)\delta t$, p_0 and p_1 being