Refer: Chapter 6, Schwinger et al.
Classical Electrodynamics.

Diamagnetism

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1. The potential energy associated with a magnetic moment $\vec{m}$ interacting with a magnetic field $\vec{B}$ is

$$U = -\vec{m}\cdot\vec{B}$$

2. We shall use the above expression to define the magnetic moment of a macroscopic system, like a solid or gas that constitutes moving charges. In particular, the magnetic moment of a macroscopic system is defined by the variational statement

$$\delta U = -\vec{m}\cdot\delta\vec{B},$$

which can be written as

$$\vec{m} = -\frac{\partial U}{\partial \vec{B}}.$$

That is, magnetic moment is the generalized 'force' associated with the system trying to minimize its energy with respect to $\vec{B}$.

Keywords:
- → Free energy in thermodynamics.
- → Virtual work.

3 The energy of a macroscopic material in a uniform magnetic field is

$$U = \sum_a \frac{\left[\vec{P}_a - q_a \vec{A}(\vec{r}_a)\right]^2}{2m_a} + q_a \phi(\vec{r}_a)$$

sum over all charges

uniform electric field we have.

4 For uniform magnetic field we

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$$

$$\phi = \vec{E} \cdot \vec{r}$$

with respect to $\vec{B}$ is

5 The charge in energy

$$\delta U = \sum_a \frac{1}{2m_a} \left[\vec{P}_a - \frac{q_a}{2} \vec{B} \times \vec{r}_a\right] \cdot 2\left[-\frac{q_a}{2} \delta\vec{B} \times \vec{r}_a\right]$$

$$= -\sum_a \frac{q_a}{2m_a} \left(\delta\vec{B} \times \vec{r}_a\right) \cdot \left[\vec{P}_a - \frac{q_a}{2} \vec{B} \times \vec{r}_a\right]$$

$$= -\delta\vec{B} \cdot \sum_a \frac{q_a}{2m_a} \vec{r}_a \times \left[\vec{P}_a - \frac{q_a}{2} \vec{B} \times \vec{r}_a\right]$$

(swapping dot and cross)

Thus,

$$\vec{m} = -\frac{\partial U}{\partial \vec{B}} = \sum_a \frac{q_a}{2m_a} \vec{r}_a \times \left[\vec{P}_a - \frac{q_a}{2} \vec{B} \times \vec{r}_a\right]$$

6 Again,

$$\vec{m} = \sum_a \frac{q_a}{2m_a} \vec{r}_a \times \vec{p}_a - \sum_a \frac{q_a^2}{4m_a} \vec{r}_a \times (\vec{B} \times \vec{r}_a)$$

7 Intrinsic magnetic moment $\vec{M}_{in}$ of a macroscopic material will be defined as $\vec{M}$ evaluated at $\vec{B} = 0$. Thus,

$$\vec{m}_{in} = \vec{m}\Big|_{\vec{B}=0}$$

$$= \sum_a \frac{q_a}{2m_a} \vec{r}_a \times \vec{p}_a$$

$\vec{L}_a = \vec{r}_a \times \vec{p}_a$

angular momentum.

8(a) So,

$$\vec{m} = \vec{m}_{in} + \vec{m}_{dia}$$

where the induced magnetic moment due to the magnetic field $\vec{B}$ is

$$\vec{m}_{dia} = - \sum_a \frac{q_a^2}{4m_a} \vec{r}_a \times (\vec{B} \times \vec{r}_a)$$

8(b) Magnetization:

$$\vec{M} = n\, \vec{m}$$

$n = \frac{\text{atoms}}{\text{volume}}$

$\vec{M} = \frac{\text{magnetic moment}}{\text{volume}}$

(9) The induced magnetic moment is a response to the magnetic field $\vec{H}$. Thus, we can write

$$\vec{M}_{dia} = \overleftrightarrow{\chi}_m \cdot \vec{H}$$

where $\overleftrightarrow{\chi}_m$ is the magnetic susceptibility of the material. To read out $\overleftrightarrow{\chi}_m$ we rewrite (8) as

$$\begin{aligned}
\vec{M}_{dia} &= -n \sum_a \frac{q_a^2}{4m_a} \left[r_a^2 \vec{B} - \vec{r}_a (\vec{r}_a \cdot \vec{B}) \right] \\
&= -n \sum_a \frac{q_a^2}{4m_a} \left[r_a^2 \overleftrightarrow{1} - \vec{r}_a \vec{r}_a \right] \cdot \vec{B} \\
&= \frac{\overleftrightarrow{\chi}_m'}{\mu_0} \cdot \vec{B}
\end{aligned}$$

where

$$\frac{\overleftrightarrow{\chi}_m'}{\mu_0} = -n \sum_a \frac{q_a^2}{4m_a} \left[r_a^2 \overleftrightarrow{1} - \vec{r}_a \vec{r}_a \right]$$

(10) Macroscopic Maxwell equations (in SI units)

$$\vec{\nabla}\cdot\left(\epsilon_0\vec{E}+\vec{P}\right)=\rho$$

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}-\vec{M}\right)=\frac{\partial}{\partial t}\left(\epsilon_0\vec{E}+\vec{P}\right)+\vec{J}$$

(11) Define:

$$\vec{D}=\epsilon_0\vec{E}+\vec{P}$$

$$\vec{H}=\frac{\vec{B}}{\mu_0}-\vec{M}$$

χ_e - electric susceptibility

χ_m - magnetic susceptibility

Linear responses:

$$\vec{P}=\epsilon_0\overleftrightarrow{\chi_e}\cdot\vec{E}$$

$$\vec{M}=\overleftrightarrow{\chi_m}\cdot\vec{H}$$

ϵ - electric permittivity

μ - magnetic permeability

Define:

$$\vec{D}=\overleftrightarrow{\epsilon}\cdot\vec{E}$$

$$\vec{B}=\overleftrightarrow{\mu}\cdot\vec{H}$$

(12) We have

$$\overleftrightarrow{\chi_e}=\frac{\overleftrightarrow{\epsilon}}{\epsilon_0}-\overleftrightarrow{1}$$

$$\overleftrightarrow{\chi_m}=\frac{\overleftrightarrow{\mu}}{\mu_0}-\overleftrightarrow{1}$$

(12) Using (9), (10) and (11)

$$\overleftrightarrow{\chi}_m \cdot \vec{H} = \frac{\overleftrightarrow{\chi}_m'}{\mu_0} \cdot \vec{B}$$

$$= \overleftrightarrow{\chi}_m' \cdot \left(\overleftrightarrow{1} + \overleftrightarrow{\chi}_m\right) \cdot \vec{H}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$$= \frac{\vec{B}}{\mu_0} - \overleftrightarrow{\chi}_m \cdot \vec{H}$$

$$\vec{B} = \mu_0 \left(\overleftrightarrow{1} + \overleftrightarrow{\chi}_m\right)$$

Thus
$$\overleftrightarrow{\chi}_m = \overleftrightarrow{\chi}_m' \cdot \left(\overleftrightarrow{1} + \overleftrightarrow{\chi}_m\right)$$

or
$$\overleftrightarrow{\chi}_m = \frac{\overleftrightarrow{\chi}_m'}{\overleftrightarrow{1} - \overleftrightarrow{\chi}_m'}$$

$$\approx \overleftrightarrow{\chi}_m' \qquad \left(\text{for } |\overleftrightarrow{\chi}_m'| << 1\right)$$

(13) Thus, for weak responses $\left(|\overleftrightarrow{\chi}_m'| << 1\right)$ we have

$$\overleftrightarrow{\chi}_m = \overleftrightarrow{\chi}_m' = -\mu_0 \, n \sum_a \frac{q_a^2}{4 m_a} \left[r_a^2 \, \overleftrightarrow{1} - \vec{r}_a \, \vec{r}_a \right].$$

(14) Summary:

$$\vec{M} = \vec{M}_{in} + \vec{M}_{dia}$$

$$\vec{M}_{in} = n \sum_a \frac{q_a}{2 m_a} \vec{r}_a \times \vec{p}_a \longrightarrow \text{intrinsic magnetic moment.}$$

$$\vec{M}_{dia} = \overleftrightarrow{\chi}_m \cdot \vec{B} \longrightarrow \text{induced magnetic moment that leads to diamagnetism.}$$

(15) Let us consider spherically symmetric atoms (completely filled shells) such that for identical $\frac{q_a}{m_a}$

$$\sum_a \frac{q_a}{2m_a} \vec{r}_a \times \vec{p}_a = \frac{q}{2m} \sum_a \vec{r}_a \times \vec{p}_a$$

$$= \frac{q}{2m} \vec{L} \longrightarrow = 0 \text{ for spherically symmetric.}$$

$$= 0.$$

(16) For this isotropic case we have

$$\overleftrightarrow{\chi}_m = -\mu_0 n \frac{q^2}{4m} \sum_a \left[r_a^2 \overleftrightarrow{1} - \vec{r}_a \vec{r}_a \right]$$

(17) The sum over charges for spherically symmetric systems is equivalent to averaging over directions.

That is,

$$\sum_a \left[r_a^2 \overleftrightarrow{1} - \vec{r}_a \vec{r}_a \right] = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \left[r^2 \overleftrightarrow{1} - \vec{r} \vec{r} \right]$$

r - radius of shell.

(18) $$\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \; \vec{r} \vec{r} = \frac{1}{3} r^2 \overleftrightarrow{1}$$

$$\Rightarrow \sum_a \left[r_a^2 \overleftrightarrow{1} - \vec{r}_a \vec{r}_a \right] = \frac{2}{3} r^2 \overleftrightarrow{1}$$

19 Using 18 in 16

$$\overleftrightarrow{\chi}_m = -\mu_0 \, n \, \frac{q^2 r^2}{6m} \overleftrightarrow{1}$$

20 → The negative sign means that the induced magnetic moment is opposite in direction of $\vec{B}$.

This is a consequence of Lenz's law.

21 Typical value:

$$\chi_m \sim -\left(4\pi \times 10^{-7} \, kg \frac{m}{C^2}\right) \frac{1}{(3\times 10^{-10} m)^3} \frac{(1.6\times 10^{-19} C)^2 (10^{-10} m)^2}{6 \times (9.1\times 10^{-31} kg)}$$

$$= -10^{-6}$$