Landau diamagnetism: A simple calculation

A. Dupré

Citation: American Journal of Physics **49**, 34 (1981); doi: 10.1119/1.12607
View online: https://doi.org/10.1119/1.12607
View Table of Contents: https://aapt.scitation.org/toc/ajp/49/1
Published by the American Association of Physics Teachers

Landau diamagnetism: A simple calculation

A. Dupré
Physics Department, Katholieke Universiteit Leuven, Celestijnenlaan 200D, B-3030 Leuven, Belgium
(Received 13 December 1979; accepted 6 June 1980)

Starting from the energy and degeneracy of the Landau levels of a free-electron gas in a magnetic field, the nonoscillatory term of the Landau diamagnetism is derived for $T = 0$, using elementary algebra only.

I. INTRODUCTION

The Pauli spin paramagnetism of conduction electrons is a subject which is treated in most courses on magnetism, metal physics, and solid state physics. Its logical counterpart, the Landau orbital diamagnetism, however, is a far less common topic in elementary texts. This is because the mathematics, even in the free-electron theory, is difficult and obscures the physics of the problem for many students. Most authors therefore illustrate the derivation with qualitative arguments that make the diamagnetism plausible, once the energy of the Landau levels and their degeneracy have been computed.

Using elementary algebra only, these arguments can be made quantitative in such a way as to yield the correct value of the nonoscillatory part of the susceptibility in the free-electron approximation at zero temperature. Depending on the level of the course, such a simple derivation can be a valuable substitute for a more complete treatment of the problem.

II. LANDAU LEVELS

We consider the conduction electrons to be contained in a rectangular parallelepiped with sides L_x, L_y, L_z and volume V. By applying periodic boundary conditions, the wave vector components are quantized:

$$k_j = 2\pi n_j / L_j \quad j = x,y,z, \tag{1}$$

and the allowed states are represented by the points of an orthorhombic lattice in k space. When a magnetic field **B** along z is applied to the free-electron gas, the points in a plane perpendicular to the k_z axis in this lattice will coalesce into the Landau levels, which are represented by concentric circles in these planes. The energy of the free electrons is now given by

$$E = (n + \tfrac{1}{2})\hbar\omega + \hbar^2 k_z^2/2m,$$

with $\omega = qB/m$ (q is the absolute value of the electron charge and B is the magnetic field). As shown in many textbooks on solid state physics, the degeneracy of the Landau levels is

$$p = m\omega L_x L_y/\pi\hbar,$$

taking spin degeneracy into account.

In zero field the density of states in a plane perpendicular to the k_z axis is easily seen to be $mL_xL_y/\pi\hbar^2$, independent of E. The number of states in an energy range $\hbar\omega$ is

$$m\omega L_x L_y/\pi\hbar = p.$$

The overall density of states is thus not affected by the magnetic field. This allows us to associate a bunch of p nonmagnetic levels with each p-fold degenerate Landau level. Each bunch of $B = 0$ levels is to be chosen in such a way that the mean energy of these levels is equal to the energy E_L of the corresponding Landau level. Since the density of states in the plane is independent of the energy, this is simply done by considering a bunch which extends from $E_L - \hbar\omega/2$ to $E_L + \hbar\omega/2$. As the first Landau level has an energy $\hbar\omega/2$, we can start with a whole bunch at the low side of the energy scale.

III. CALCULATION OF THE ENERGY FOR $B = 0$

Since the mean energy of a bunch of $B = 0$ levels coincides with the $B \neq 0$ Landau level, and since the Landau level can contain the same number of electrons as the bunch, the energy of a completely filled bunch will not be affected by applying the field. Therefore one must only consider these bunches of levels which are intersected by the Fermi surface. Because we make a ground-state calculation, we assume for these top bunches that the part below the Fermi level is completely filled, and that the part above it is completely empty.

The component $k^i_{\max}$ perpendicular to the field, of a wave vector of a point on the Fermi sphere in the ith plane obeys (Fig. 1)

$$k_F^2 = (k^i_{\max})^2 + (k_z^i)^2$$

or

$$E_F = E_{\max}(k_z^i) + \hbar^2(k_z^i)^2/2m \tag{2}$$

in which $\hbar^2(k_z^i)^2/2m$ is the energy due to the motion along the field and $E_{max}(k_z^i)$ is the maximum energy due to the motion perpendicular to the field for a particular value of k_z^i.

Let E_i be the Landau level corresponding with the partially occupied bunch for k_z^i (Fig. 2). Since the density of states in each plane is independent of the energy, the mean energy (due to the motion perpendicular to the field) of the electrons in the top bunch is the arithmetic mean of the maximum energy $E_{\max}$ and the bottom energy $E_i - \hbar\omega/2$, i.e.,

$$(1/2)(E_{\max} + E_i - \hbar\omega/2).$$

For the same reason the occupation of a partially filled bunch is equal to the occupation p of a completely filled bunch times the relative energy range over which the occupied part of the bunch extends, i.e.,

$$p[E_{\max} - (E_i - \hbar\omega/2)]\hbar\omega.$$

With $x_i = E_{\max} - E_i$, the energy (due to the motion perpendicular to the field) of the electrons in one partially

 0002-9505/81/010034-03$00.50

occupied bunch is

$$(p/2)[E_{\max} - \hbar\omega/4 + x_i(2E_{\max}/\hbar\omega - 1) - x_i^2/\hbar\omega]. \tag{3}$$

We shall assume that the values of x_i are randomly distributed in the range between $-\hbar\omega/2$ and $\hbar\omega/2$. This will be a good approximation if the number of Landau levels below $E_{\max}$ is large and if $E_{\max}$ varies rapidly with k_z.

Averaging over all the planes, we have

$$\langle x_i \rangle = 0, \tag{4a}$$

$$\langle x_i^2 \rangle = \frac{1}{\hbar\omega} \int_{-\hbar\omega/2}^{\hbar\omega/2} x^2 dx = \hbar^2\omega^2/12. \tag{4b}$$

From $\langle x_i \rangle = 0$ it follows that the mean occupation of the top bunches is $p/2$, so that the mean energy of its electrons due to their motion along the field is given by

$$(p/2)[\hbar^2(k_z^i)^2/2m]. \tag{5}$$

The mean value of the total energy of the electrons in a top bunch is the sum of (3) and (5). With (2) and (4) this value can be written as

$$(p/2)(E_F - \hbar\omega/3).$$

From (1) it follows that the number of k_z^i values is $k_F L_z/\pi$. The resulting total energy of the top bunches is therefore

$$U(B = 0) = (k_F L_z/\pi)(p/2)(E_F - \hbar\omega/3).$$

IV. CALCULATION OF THE ENERGY FOR $B \neq 0$

If the k_z value of the electrons would be unchanged when the magnetic field is switched on, the energy of all the electrons of a top bunch would become E_i, the energy of the corresponding Landau level. Since the mean energy E_i of the whole bunch is larger than the mean energy of the occupied part of the bunch in zero field, the energy is seen to increase with the field. The value of the magnetic susceptibility computed from this energy increase is twice the true value. This is because the configuration with unchanged k_z is not the ground state. Indeed, since x_i is supposed to be randomly distributed between $\hbar\omega/2$ and $\hbar\omega/2$, there will be as many partially occupied Landau levels above the Fermi surface as below. For the same reason the number of electrons on partially occupied Landau levels outside the Fermi sphere is equal to the number of vacant states on the uncompletely occupied Landau levels below the Fermi surface. If the electrons are allowed to be redistributed among the different k_z^i, all the levels with $E_i > E_{\max}$ can be emptied and all the levels with $E_i < E_{\max}$ completely (p) occupied.[1] Again because of the random distribution of x_i, the overall momentum and energy due to the motion in the z direction will be unchanged.

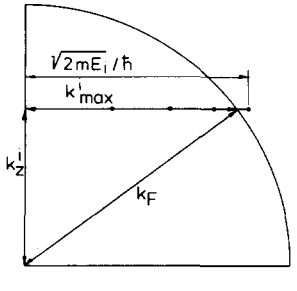

Fig. 1. Illustration of the relation between k_F, k_z^i, and $k_{\max}^i$. Landau levels are indicated by dots.

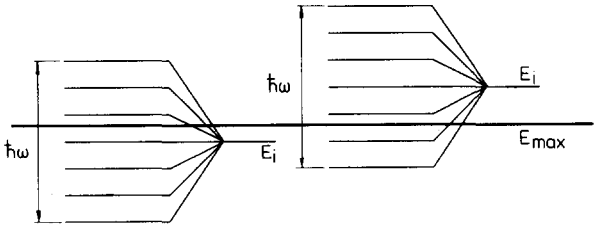

Fig. 2. Energy due to the motion perpendicular to the field. Two possible positions of the top bunch with respect to $E_{\max}$ are shown.

The energy for $B \neq 0$ of the electrons in the Landau levels E_i is now simply $pE_i = p(E_{\max} - x_i)$ for $x_i > 0$ and zero for $x_i < 0$. The mean value of x_i for $x_i > 0$ is $\hbar\omega/4$. Because we assumed an equal number of $x_i > 0$ and $x_i < 0$, the average energy due to the motion perpendicular to the field is now

$$(p/2)(E_{\max} - \hbar\omega/4).$$

The resulting total energy of the electrons in the Landau levels E_i is

$$U(B \neq 0) = (k_F L_z/\pi)(p/2)(E_F - \hbar\omega/4).$$

V. LANDAU DIAMAGNETISM

The energy increase due to the field is

$$\begin{aligned}\Delta U &= U(B \neq 0) - U(B = 0) \\ &= (pk_F L_z/2\pi)(\hbar\omega/3 - \hbar\omega/4) \\ &= pk_F L_z \hbar\omega/24\pi.\end{aligned}$$

Substituting the expression for p and ω, and putting $L_x L_y L_z = V$, we have

$$\Delta U = Vk_F q^2 B^2/24\pi^2 m.$$

The susceptibility is defined as $\chi = -(\mu_0/V)(\partial^2 F/\partial B^2)$. Since $\partial F/\partial B = \partial(\Delta U)/\partial B$, one gets for the susceptibility

$$\chi = -\mu_0 k_F q^2/12\pi^2 m.$$

This is the result derived by Landau.[2]

We must come back to the assumption concerning the random distribution of x_i. The number of Landau levels below the Fermi surface is large for the majority of k_z^i values if $\hbar\omega \ll E_F$. This condition is very well fulfilled for all man-made magnetic fields and for ordinary metals.

The condition that $E_{\max}$ varies strongly with k_z is not fulfilled for small $|k_z|$, i.e., near the equator of our spherical Fermi surface. For $kT \ll \hbar\omega$, and thus certainly for the $T = 0$ case which we consider here, there will be a term in the energy which varies periodically with B. This term gives rise to the de Haas–van Alphen effect and other oscillatory phenomena. Another derivation of the Landau result has been given by Pippard.[3,4] This author considers the periodic susceptibility due to the electrons in the top bunches for all k_z, and not only for small $|k_z|$. The Landau susceptibility is then obtained as the mean value over one period of oscillation. So far this author's treatment is of comparable mathematical simplicity. In order to show that the oscillatory part of the susceptibility comes from the electrons with small $|k_z|$ and that the field independent Landau susceptibility comes mainly from the electrons for which $|k_z|$ is

not very small, Pippard refers to some more involved mathematics. The latter can be avoided in the present treatment.

VI. CONCLUSION

We have derived the Landau value of the orbital magnetic susceptibility of conduction electrons in the free electron approximation at $T = 0$. We think that the restrictions of this calculation are compensated, for pedagogical purposes, by the simplicity of the mathematics used and consequently, by the possibility of a better understanding of the physics of the problem.

[1]Among the few books in which this redistribution is emphasized, we mention F. Seitz, *The Modern Theory of Solids* (McGraw-Hill, New York, 1940), p. 587.

[2]L. Landau, Z. Phys. **64**, 629 (1930).

[3]A. B. Pippard, in *Low Temperature Physics*, 1961 Session of the Les Houches Summer School, edited by C. De Witt, B. Dreyfus, and P. G. de Gennes (Gordon and Breach, New York, 1962), p. 14–23.

[4]A. B. Pippard, Rep. Prog. Phys. **23**, 206–210 (1960).