3 Landu Diamagnetism

The classical and quantum version of perevious discussion explain the diamagnetism contributed due to the bound electron. There is still lalking of diamagnetism in free electron gas (i.e. metal). Landau explain the diamagnetism for free electron system by using the quantum statistical approach.

The Hamiltonian of a nonrelativistic electron in external magnetic field.

$$\mathcal{H} = \frac{(\vec{P} + e/c\, \vec{A})^2}{2m} \quad \ldots \quad (3.1) \qquad (e \text{ is positive})$$

we know $\vec{A} = \vec{B} \times \vec{r}$ if applied field H is along the z direction than $A_x = -By$, $A_y = Bx$, $A_z = 0$

Than

$$\mathcal{H} = \frac{(P_x - e/c\, Hy)^2}{2m} + \frac{(P_y + e/c\, Hx)^2}{2m} + \frac{P_z^2}{2m} \quad (3.2)$$

Let's apply the gauge transformation under which schrodinger equation $\mathcal{H}\psi = E\psi$ invariant

$$\vec{A}(\vec{r}) \to \vec{A}(\vec{r}) - \nabla\omega(r)$$

$$\psi(r) \to \exp\left[-\frac{ie}{\hbar c}\omega(r)\right]\psi(r)$$

So, $A_y = 0$

Now, the Hamiltonian (3.2) becomes

$$\mathcal{H} = \frac{(P_x - e/c\, Hy)^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} \quad (3.3)$$

we can solve the schrödinger wave equation by assuming wave function of the form

$$\psi(x, y, z) = e^{i(K_x x + K_z z)} f(y) \quad \ldots (3.4)$$

Now, using

$$\mathcal{H}\Psi = E\Psi$$

$$\frac{1}{2m}\left\{ \left(p_x - \tfrac{e}{c}Hy\right)^2 + p_y^2 + p_z^2 \right\} e^{i(k_x x + k_z z)} f(y) = \epsilon e^{i(k_x x + k_z z)} f(y)$$

$$\frac{1}{2m}\left\{ \left(-i\hbar \partial/\partial x - \tfrac{e}{c}Hy\right)^2 + p_y^2 + \left(-i\hbar \partial/\partial z\right)^2 \right\} e^{i(k_x x + k_z z)} f(y) = \epsilon e^{i(k_x x + k_z z)} f(y)$$

$$\frac{1}{2m}\left\{ i^2 i^2 k_x^2 \hbar^2 + 2i^2 \frac{\hbar e}{c} H y k_x + \frac{e^2}{c^2} H^2 y^2 + p_y^2 + i^2 i^2 k_z^2 \hbar^2 \right\} e^{i(k_x x + k_z z)} f(y) = \epsilon e^{i(k_x x + k_z z)} f(y)$$

$$\frac{1}{2m}\left[\left\{ \hbar^2 k_x^2 - \frac{2\hbar e H y k_x}{c} + \frac{e^2 H^2 y^2}{c^2} \right\} + p_y^2 \right] f(y) = \left\{\epsilon - \frac{k_z^2 \hbar^2}{2m}\right\} f(y).$$

$$\left[\left\{ \frac{\hbar^2 k_x^2}{2m} - \frac{2\hbar e H y k_x}{2mc} + \frac{e^2 H^2 y^2}{2mc^2} \right\} + p_y^2/2m \right] f(y) = \epsilon' f(y)$$

$$\left[p_y^2/2m + \frac{1}{2} m \left(\frac{eH}{mc}\right)^2 \left\{ y^2 - 2y\left(\frac{\hbar c}{eH}\right) k_x + \frac{\hbar^2 c^2 k_x^2}{e^2 H^2} \right\} \right] f(y) = \epsilon' f(y) \quad \ldots (3.5)$$

$$\left[p_y^2/2m + \tfrac{1}{2} m \omega_0^2 \left\{ (y - y_0)^2 \right] f(y) = \epsilon' f(y) \quad (3.6)$$

where, $\omega_0 = eH/mc$, $y_0 = \hbar c/eH \; k_x$

and $\epsilon' = \epsilon - \hbar^2 k_z^2/2m$

The function $f(y)$ satisfies the equation for a harmonic oscillator with frequency $\omega_0 = eH/mc$ ("cyclotron frequency"). ~~that of~~

Thus energy eigen value $\epsilon' = \hbar\omega_0(j + 1/2)$ for Harmonic oscillator

$$\epsilon - \frac{\hbar^2 k_z^2}{2m} = \hbar\omega_0(j + 1/2)$$

$$\epsilon = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega(j+1/2) \quad \ldots (3.7)$$

where $j = 0, 1, 2 \ldots$

and $p_z = \hbar k_z$

$$\epsilon(p_z, j) = \frac{p_z^2}{2m} + \hbar\omega_0(j+1/2) \quad \ldots (3.8)$$

There are the landu levels. These energy eigen states are independent of k_x. They have a degeneracy equal to the number of allowed values of k_x such that y_0 lies with in the container of the system.

If we put the system in a large cube of size L. The periodic boundary condition allowed the k_x value of $\frac{2\pi n_x}{L}$ where $n_x = 0, \pm 1, \pm 2 \ldots$

If y_0 lies between 0 & L.

The values of n_x must be positive

$$y_0 = \left(\frac{\hbar c}{eH}\right)\frac{2\pi n_x}{L}$$

$$y_0 = L, \quad n_x = g$$

$$L = \left(\frac{\hbar c}{eH}\right)\frac{2\pi g}{L}$$

$$L = \frac{h}{2\pi}\frac{c}{eH}\frac{2\pi g}{L}$$

$$g = \left(\frac{eH}{hc}\right)L^2 \quad \ldots (3.9)$$

~~$g = \left(\frac{hc}{eH}\right)L^2 \quad \ldots (3.9)$~~

which is the degeneracy of the landau level.

Now ~~using~~ the grand partition function is

$$\mathcal{Z} = \prod_{\lambda} (1 + z e^{-\beta \epsilon_{\lambda}}) \quad \ldots \quad (3.10)$$

where λ denotes the set of quantum numbers $\{p_z, j, \alpha\}$ with $\alpha = 1 \ldots g$. Thus.

$$\log \mathcal{Z} = \sum_{\alpha=1}^{\infty} \sum_{j=0}^{\infty} \sum_{p_z} \log [1 + z e^{-\beta \epsilon(p_z, j)}]$$

$$= 2g \sum_{j=0}^{\infty} \frac{L}{h} \int_0^{\infty} dp \log [1 + z e^{-\beta \epsilon(p, j)}]$$

$$\sum_{p_z} \log [1 + z e^{-\beta \epsilon(p_z, j)}] = \frac{L}{h} \int_0^{0} dp \log (1 + z e^{-\beta \epsilon}$$

For one-dimension, $p_z \to p$

for the maximum value $\sum_{\alpha=1}^{g} = 2g$.

~~$\log \mathcal{Z} = \frac{2gL}{h} \sum_{j=0}^{\infty} \int_0^{\infty} dp \, 1$~~

$$\log \mathcal{Z} = \frac{2gL}{h} \sum_{j=0}^{\infty} \int_0^{\infty} dp \log [1 + z e^{-\beta \epsilon(p, j)}] \quad \ldots \quad (3.11)$$

The average number of electron is

$$N = z \frac{\partial}{\partial z} \log \mathcal{Z} = z \frac{2gL}{h} \sum_{j=0}^{\infty} \int_0^{\infty} dp \frac{e^{-\beta \epsilon(p, j)}}{1 + z e^{-\beta \epsilon(p, j)}}$$

$$N = \frac{2gL}{h} \sum_{j=0}^{\infty} \int_0^{\infty} \frac{dp}{z^{-1} e^{\beta \epsilon(p, j)} + 1} \quad \ldots \quad (3.12)$$

The magnetization in classical domain will takes place at high temperature. The average number of electron 'N' is finite in the limit $z \to 0$ at eq^n (3.11)

Let's expand the eq^n(3.11) in the power of z and ~~retain~~ ~~the first order term~~ neglect the higher order term.

$$\text{Log } Z_e = \frac{2gL}{h} \sum_{j=0}^{\infty} \int_0^{\infty} dp \, z e^{-\beta \epsilon(p,j)} \qquad \left[\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \right.$$

$$\text{Log } Z_e = \frac{2zgL}{h} \sum_{j=0} \int_0^{\infty} dp \, e^{-\beta [p^2/2m + \hbar\omega_0 (j+1/2)]} \quad [\because \text{from eq}^n (3.8)]$$

$$\text{Log } Z_e = \frac{2zgL}{h} \int_0^{\infty} dp \, e^{-\frac{\beta p^2}{2m}} \left[e^{-\frac{\beta\hbar\omega_0}{2}} \{ 1 + e^{-\beta\hbar\omega_0} + e^{-2\beta\hbar\omega_0} + e^{-3\beta\hbar\omega_0} + \cdots \right.$$

$$\text{Log } Z_e = \frac{2gzL}{h} \int_0^{\infty} dp \, e^{-\beta p^2/2m} \, e^{-x} [1 + e^{-2x} + e^{-4x} + e^{-6x} + \cdots]$$

$$\text{where } x = \frac{\beta\hbar\omega_0}{2} = \frac{\hbar\omega_0}{2KT}$$

~~$1 + e^{-2x} + e$~~

$$1 + (e^{-2x})^1 + (e^{-2x})^2 + (e^{-2x})^3 + \cdots = \frac{1}{1 - e^{-2x}}$$

~~$1+(e^{-2x})$~~

$$\int_0^{\infty} dp \, e^{-\frac{p^2}{2KTm}} = \frac{1}{2\lambda} \quad \text{where } \lambda = \sqrt{\frac{2\pi\hbar^2}{mKT}}, \quad \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\pi/a}$$

$$(p = \hbar K_z)$$

$$\text{Log } Z_1 = \frac{zgL}{\lambda} \frac{e^{-x}}{1 - e^{-2x}} \quad \cdots (3.13)$$

Again re-arranging the eqn (3.13)

$$\log Z = \frac{zgL}{\lambda}\left(\frac{1}{e^{x}-e^{-x}}\right)$$

$$e^{x} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$
$$e^{x} = 1+x+x^2/2+x^3/6+\cdots$$
$$e^{-x} = 1-x+x^2/2-x^3/6+\cdots$$

$$\log Z = \frac{zgL}{\lambda}\,\frac{1}{(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots)-\{1-x+\frac{1}{2}x^2-\frac{1}{6}x^3\}}$$

$$\log Z = \frac{zgL}{\lambda}\,\frac{1}{2x+2x^3/6}$$

$$\log Z = \frac{zgL}{\lambda}\,\frac{1}{2x}\,\frac{1}{(1+x^2/6)}$$

$$\log Z = \frac{zgL}{\lambda}\,\frac{1}{2x}\,(1+x^2/6)^{-1}$$

expanding the term and neglecting higher terms

$$\log Z = \frac{zgL}{\lambda}\,\frac{1}{2x}\,(1-x^2/6) \quad \cdots (3.14)$$

and from the eqn (3.9)

we have $x = \frac{\hbar\omega_0}{2KT}$, $\omega_0 = eH/mc$

$$g = (eH/ch)L^2$$

Then

$$g/2x = \frac{(eH/ch)L^2}{2\,\frac{\hbar\omega_0}{2KT}} = \frac{(eH/ch)L^2\,KT}{\hbar\,eH/mc}$$

$$g/2x = \left(\frac{mKT}{2\pi\hbar^2}\right)L^2 = \frac{L^2}{\lambda^2}$$

$$\log Z_1 = \frac{z L^3}{\lambda^3}\left(1 - \frac{x^2}{6}\right)$$

$$\log Z = \frac{zV}{\lambda^3}\left\{1 - \frac{1}{6}\left(\frac{\hbar\omega_0}{2KT}\right)^2\right\}$$

$$\log Z = \frac{zV}{\lambda^3}\left\{1 - \frac{1}{24}\left(\frac{\hbar\omega_0}{KT}\right)^2\right\} \quad \ldots 3.15$$

$$\log Z = \frac{zV}{\lambda^3}\left\{1 - \frac{1}{24}\left(\frac{e\hbar}{mcKT}\right)^2 H^2\right\} \quad 3.16.$$

we also, know that to eliminate z, from (3.11)

$$N = \log Z$$

$$N = \frac{zV}{\lambda^3} \quad \text{i.e} \quad \frac{N}{V} = \frac{z}{\lambda^3}$$

$$\log Z = \frac{N}{V}$$

From the definition of induced magnetic moment per unit volume of the system along the direction of external magnetic field H, in grand canonical ensemble is given by

$$M = KT \frac{\partial}{\partial H}\left(\frac{\log Z}{V}\right)_{T,V,z} \quad \ldots 3.17$$

The magnetic susceptibility per unit volume of the system is defined to be

$$\chi = \frac{\partial M}{\partial H} \quad (3.18)$$

From eqn (3.18) & (3.17)

$$\chi = KT \frac{\partial^2}{\partial H}\left\{\frac{\log Z}{V}\right)_{T, V, z} \quad \ldots (3.19)$$

Now, let use eqn (3.16) in 3.19

$$\chi_1 = KTz \frac{\partial^2}{\partial H^2}\left\{\frac{zV}{V\lambda^3}\left[1 - \frac{1}{24}\left(\frac{e\hbar}{cmKT}\right)^2 H^2\right\}\right.$$

$$\chi = KT\left(-\frac{1}{12}\right)\frac{e^2\hbar^2}{m^2K^2T^2c^2}$$

$$\chi = -\frac{z}{3KT\lambda^3}\left(\frac{e\hbar}{2mc}\right)^2 \quad \ldots (3.20)$$

To eliminate z, we will get (3.11) to the first order in z

so $N = \log Z$

ie $N/V \sim \frac{z}{\lambda^3}$

$$v = \frac{z}{\lambda^3}$$

Now eqn (3.20) becomes

$$\boxed{\chi = -\frac{1}{3KTv}\left(\frac{e\hbar}{2mc}\right)} \quad (3.21).$$

which is conformis to curie's $1/T$ law.