# Addition of spin-1/2 particles

Let us first review quantum mechanics of a single spin 1/2 particle

Assuming that the $S^2$ and $S_z$ are good quantum ~~so~~ numbers

~~(crossed)~~ since, $[S^2, S_z] = 0$ but $[S_x, S_y] = i\hbar S_z$ and so on ..

we cannot find simulteneous eigenstates of $S^2, S_x, S_y, S_z$

we pick $S^2$ and $S_z$.

For spin 1/2 particles $s = 1/2$ and $S_z = \pm 1/2$

So, we will denote $S = 1/2$ and $S_z = +1/2$ state by $|\uparrow\rangle$

$$|S, S_z\rangle \triangleq \left|\tfrac{1}{2}, +\tfrac{1}{2}\right\rangle = |\uparrow\rangle \;;\; \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = |\downarrow\rangle$$

$$S^2|\uparrow\rangle = s(s+1)\hbar^2|\uparrow\rangle = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\hbar^2|\uparrow\rangle = \tfrac{3}{4}\hbar^2|\uparrow\rangle$$

$$S^2|\downarrow\rangle = \tfrac{3}{4}\hbar^2|\downarrow\rangle \;;\; S_z|\uparrow\rangle = +\tfrac{\hbar}{2}|\uparrow\rangle$$
$$S_z|\downarrow\rangle = -\tfrac{\hbar}{2}|\downarrow\rangle$$

In Matrix notation
$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$S^2 = \tfrac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \;;\; S_z = \tfrac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

what about $S_x$ and $S_y$?

we introduce them in terms of the ladder operators

$S_\pm = S_x \pm i S_y$ where $S_+$ increases $S_z$ by $+1$ and $S_-$ lowers $S_z$ by $-1$. But since $S_z = \pm 1/2$ only ...

ie $S_+|\uparrow\rangle = 0$ and $S_-|\downarrow\rangle = 0$

But $S_+|\downarrow\rangle = c_+|\uparrow\rangle$ and $S_-|\uparrow\rangle = c_-|\downarrow\rangle$

$\boxed{S_\pm \text{ does not change } s, \text{ only } S_z.}$

In general $S_\pm|s, S_z\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\,|s, S_z \pm 1\rangle$

In matrix notation:— $S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ; $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

So, specifically $S_+|\downarrow\rangle = \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1)-(-\frac{1}{2})(-\frac{1}{2}+1)}\,|\uparrow\rangle$

$= \hbar\sqrt{\frac{3}{4}+\frac{1}{4}} = \hbar|\uparrow\rangle$

and $S_-|\uparrow\rangle = \hbar\sqrt{3/4 - \frac{1}{2}(\frac{1}{2}-1)}\,|\downarrow\rangle = \hbar\sqrt{3/4+1/4}\,|\downarrow\rangle = \hbar|\downarrow\rangle$

Lets proceed to addition of two spin 1/2 particles

we can use the classical picture to add spins

consider spins $\bar{S}_1$ and $\bar{S}_2$ of two particles

Total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ If $|\vec{S}_1| = |\vec{S}_2|$ we have

two max. values are $\vec{S} = \bar{S}_1 - \bar{S}_2$ or $\vec{S} = \vec{S}_1 + \vec{S}_2$

For spin 1/2 particles $|\vec{S}| = \frac{1}{2} - \frac{1}{2}$ or $\frac{1}{2}+\frac{1}{2} \Rightarrow 0$ or $1$

The two particles have their own hilbert spaces. So the spin wave function of their combination can be a product

$|S\rangle = |S,S_z\rangle_1 \otimes |S,S_z\rangle_2$ (we will drop the outer pdt notation)

There can be four possible combination based on $S_z = \pm\frac{1}{2}$

$|\frac{1}{2},\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle = |\uparrow\uparrow\rangle$

$|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle = |\downarrow\downarrow\rangle$

$|\frac{1}{2},+\frac{1}{2}\rangle|\frac{1}{2},-\frac{1}{2}\rangle = |\uparrow\downarrow\rangle$

$|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},+\frac{1}{2}\rangle = |\downarrow\uparrow\rangle$

These are the four basis states of the 2 spin 1/2 system

# Symmetric and Antisymmetric wave function by adding 2 spin-1/2 particles

| | $S$ | $S_z$ | |
|---|---|---|---|
| $\lvert\uparrow\uparrow\rangle$ | 1 | 1 | Triplet state $\lvert 1\,1\rangle, \lvert 1\,{-1}\rangle$ |
| $\lvert\downarrow\downarrow\rangle$ | 1 | -1 | $(S=1)$ $\lvert 1,0\rangle$ |
| $\dfrac{\lvert\uparrow\downarrow\rangle+\lvert\downarrow\uparrow\rangle}{\sqrt{2}}$ | 1 | 0 | |
| $\dfrac{\lvert\uparrow\downarrow\rangle-\lvert\downarrow\uparrow\rangle}{\sqrt{2}}$ | 0 | 0 | Singlet state $\lvert 0,0\rangle$ $(S=0)$ |

Lets show these explicitly . . .

Note: $S=1 \Rightarrow S^2 = 1(1+1)\hbar^2 \Rightarrow 2\hbar^2$

$$S^2 = (\vec{S}_1+\vec{S}_2)^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\bar{S}_1\cdot\bar{S}_2$$

$$2\bar{S}_1\cdot\bar{S}_2 = 2S_{1x}S_{2x} + 2S_{1y}S_{2y} + 2S_{1z}S_{2z}$$

$$S_{1x}S_{2x} + S_{1y}S_{2y} = \frac{S_1^+S_2^- + S_1^-S_2^+}{2}$$

$$\Rightarrow 2\bar{S}_1\cdot\bar{S}_2 = S_1^+S_2^- + S_1^-S_2^+ + 2S_{1z}S_{2z}$$

$\boxed{\lvert\uparrow\uparrow\rangle}$ $\quad S_1^2\lvert\uparrow\uparrow\rangle = \frac{3}{4}\hbar^2\lvert\uparrow\uparrow\rangle \quad S_2^2\lvert\uparrow\uparrow\rangle = \frac{3}{4}\hbar^2\lvert\uparrow\uparrow\rangle$

$$2\bar{S}_1\cdot\bar{S}_2 = \underbrace{\cancel{S_1^+S_2^-\lvert\uparrow\uparrow\rangle}}_{0} + \underbrace{\cancel{S_1^-S_2^+\lvert\uparrow\uparrow\rangle}}_{0} + \underbrace{2S_{1z}S_z\lvert\uparrow\uparrow\rangle}_{=\,2\times\frac{1}{2}\times\frac{1}{2}\hbar^2\lvert\uparrow\uparrow\rangle}$$

since $S_1^+\lvert\uparrow\rangle = 0$ & $S_2^+\lvert\uparrow\rangle = 0$

$$\Rightarrow S^2\lvert\uparrow\uparrow\rangle = (3/2 + 1/2)\hbar^2\lvert\uparrow\uparrow\rangle = 2\hbar^2\lvert\uparrow\uparrow\rangle$$

$\boxed{|\downarrow\downarrow\rangle}$ $\quad S_1^2|\downarrow\downarrow\rangle = S_2^2|\downarrow\downarrow\rangle = \frac{3}{4}\hbar^2|\downarrow\downarrow\rangle$

$$2S_1\cdot S_2 = \underbrace{S_1^+S_2^-|\downarrow\downarrow\rangle + S_1^-S_2^+|\downarrow\downarrow\rangle}_{0} + 2S_{1z}S_{2z}|\downarrow\downarrow\rangle$$

$$2S_{1z}S_{2z}|\downarrow\downarrow\rangle = 2\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\hbar^2 = \tfrac{1}{2}\hbar^2$$

$$\Rightarrow S^2|\downarrow\downarrow\rangle = \left(\tfrac{3}{2} + \tfrac{1}{2}\right)\hbar^2|\downarrow\downarrow\rangle$$

$\boxed{\dfrac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{2}} = |10\rangle$ we will use both notation.

$$S_1^2|\uparrow\downarrow\rangle = S_2^2|\uparrow\downarrow\rangle = \frac{3}{4}\hbar^2|\uparrow\downarrow\rangle$$

Similarly $S_1^2|\downarrow\uparrow\rangle = S_2^2|\downarrow\uparrow\rangle = \frac{3}{4}\hbar^2|\downarrow\uparrow\rangle$

So, $S_1^2\left(\dfrac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\right) = \dfrac{3}{4}\hbar^2\left(\dfrac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\right)$

$S_2^2\left(\dfrac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\right) = \dfrac{3}{4}\hbar^2\left(\dfrac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\right)$

$(S_1^2 + S_2^2)|10\rangle = \frac{3}{2}\hbar^2|10\rangle$

$$\bar{S}_1\cdot\bar{S}_2|\uparrow\downarrow\rangle = S_1^+S_2^-|\uparrow\downarrow\rangle + S_1^-S_2^+|\uparrow\downarrow\rangle + 2S_{1z}S_{2z}|\uparrow\downarrow\rangle$$
$$= 0 + \hbar^2|\downarrow\uparrow\rangle + 2\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\hbar^2|\uparrow\downarrow\rangle$$
$$= \hbar^2|\downarrow\uparrow\rangle - \frac{\hbar^2}{2}|\uparrow\downarrow\rangle$$

$$\bar{S}_1\cdot\bar{S}_2|\downarrow\uparrow\rangle = S_1^+S_2^-|\downarrow\uparrow\rangle + S_1^-S_2^+|\downarrow\uparrow\rangle + 2S_{1z}S_{2z}|\downarrow\uparrow\rangle$$
$$= \hbar^2|\uparrow\downarrow\rangle + 0 + 2\left(-\tfrac{1}{2}\right)\left(+\tfrac{1}{2}\right)|\downarrow\uparrow\rangle$$
$$= \hbar^2|\uparrow\downarrow\rangle - \frac{\hbar^2}{2}|\downarrow\uparrow\rangle$$

$$2\bar{S}_1\cdot\bar{S}_2\left(\frac{|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle}{\sqrt{2}}\right) = \hbar^2\left(\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\right) - \frac{\hbar^2}{2}\left(\frac{|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle}{\sqrt{2}}\right)$$
$$= \frac{\hbar^2}{2}|10\rangle$$

$$\Rightarrow S^2|10\rangle = \tfrac{3}{2}\hbar^2|10\rangle + \tfrac{\hbar^2}{2}|10\rangle = 2\hbar^2|10\rangle \Rightarrow \boxed{S = 1}$$

$$\boxed{\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}} = |00\rangle$$

$$S_1^2|00\rangle = \frac{3}{4}\hbar^2\frac{|\uparrow\downarrow\rangle}{\sqrt{2}} - \frac{3}{4}\hbar^2\frac{|\downarrow\uparrow\rangle}{\sqrt{2}} = \frac{3}{4}\hbar^2|00\rangle$$

Similarly $S_2^2|00\rangle = \frac{3}{4}\hbar^2|00\rangle$

$$2S_1 \cdot S_2\left(\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\right) = \left[\hbar^2\frac{|\downarrow\uparrow\rangle}{\sqrt{2}} - \frac{\hbar^2}{2}\frac{|\uparrow\downarrow\rangle}{\sqrt{2}}\right]$$

$$- \left[\hbar^2\frac{|\uparrow\downarrow\rangle}{\sqrt{2}} - \frac{\hbar^2}{2}\frac{|\downarrow\uparrow\rangle}{\sqrt{2}}\right]$$

$$= -\hbar^2\left[\frac{|\uparrow\downarrow\rangle}{\sqrt{2}} - \frac{|\downarrow\uparrow\rangle}{\sqrt{2}}\right] - \frac{\hbar^2}{2}\left[\frac{|\uparrow\downarrow\rangle}{\sqrt{2}} - \frac{|\downarrow\uparrow\rangle}{\sqrt{2}}\right]$$

$$= -\frac{3\hbar^2}{2}|00\rangle$$

$$\Rightarrow S^2 = \underbrace{\frac{3}{4}\hbar^2|00\rangle}_{S_1^2} + \underbrace{\frac{3}{4}|00\rangle}_{S_2^2} - \underbrace{\frac{3}{2}\hbar^2|00\rangle}_{2S_1 \cdot S_2} = 0$$

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Also note $S^- = S_1^- + S_2^-$

$$S^-|\uparrow\uparrow\rangle = (S_1^- + S_2^-)|\uparrow\uparrow\rangle = |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle \rightarrow |10\rangle \text{ state.}$$

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Now the $s=1$ (Triplet) state and $s=0$ (Singlet) are not degenerate and will have different energies say $E_s$ and $E_t$.

We want to find the Hamiltonian which generates the triplet and singlet states as eigenvectors with $E_t$ & $E_s$