Chapter 16
Fall of Casimir Energies in Non-commutative Space-Time

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Abstract We examine how Casimir energies gravitate in κ-deformed space-time. We set up the Casimir plates in a gravitational field using κ-deformed Rindler coordinates and compute the total force acting on the Casimir apparatus in a weak gravitational field. We show that the equivalence principle is satisfied for this setup in κ-deformed space-time.

16.1 Introduction

Constructing a consistent quantum theory of gravity is an active area of research in physics. Various approaches have been taken to construct a consistent theory of quantum gravity. All these approaches exhibit a fundamental length scale. Non-commutative geometry [1] is an approach that incorporates this fundamental length scale naturally [2].

An extensively investigated non-commutative space-time is the κ-deformed space-time [3], whose coordinates satisfy

$$[\hat{x}^i, \hat{x}^j] = 0, \quad [\hat{x}^0, \hat{x}^i] = iax^i, \quad a = \frac{1}{\kappa}, \tag{16.1.1}$$

where a is the deformation parameter with the dimension of length. The underlying space-time of the deformed special relativity is also the κ-space-time [4].

One well-studied physical phenomenon where length scale plays an important role is the Casimir effect [5]. Casimir effect and its implications have been studied in various contexts in commutative and non-commutative space-times. In [6], it is shown

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A. Giri and R. Mohanta (eds.), *Proceedings of the 17th International Conference on Interconnections Between Particle Physics and Cosmology*, Springer Proceedings in Physics 322, https://doi.org/10.1007/978-981-96-4986-0_16

that the Casimir energy reacts to the force in accordance with the equivalence principle. Here, we investigate how this result fares in the κ-deformed non-commutative space-time [7].

16.2 κ-Deformed Space-Time and Deformation of Metric

The κ-deformed space-time is a Lie-algebraic non-commutative space-time whose coordinates satisfy Eq. (16.1.1). These coordinates are realized in terms of x_i and $p_i = i\frac{\partial}{\partial x_i}$ as [8]

$$\hat{x}_0 = x_0\psi(A) + iax_j\partial_j\gamma(A), \quad \hat{x}_i = x_i\varphi(A), \tag{16.2.1}$$

where $A = ia\partial_0 = ap^0$. ψ, γ and φ are functions of A satisfying conditions $\psi(0) = 1$, $\varphi(0) = 1$, $\frac{d\varphi}{dA}\frac{\psi(A)}{\varphi(A)} = \gamma(A) - 1$. Two possible realisations of $\psi(A)$ are $\psi(A) = 1$ and $\psi(A) = 1 + 2A$ [8]. Here onwards, we choose $\psi(A) = 1$. In this study we use the $\varphi = e^{-\frac{A}{2}}$ realization.

In this study, we examine the fall of the Casimir apparatus in a uniform gravitational field in κ-space-time. Thus, we consider Rindler space-time [9] as the background, where $x \equiv (\tau, x, y, \xi)$ represents the coordinates. The proper acceleration of the particle in Rindler space-time is given by $\frac{1}{\xi}$. Background metric of the Rindler space-time is $g^{(0)}_{\mu\nu} = \text{diag}(-\xi^2, +1, +1, +1)$. Thus employing the formalism of calculating modified metric (see [10]), we find the κ-deformed Rindler metric as [7]

$$\hat{g}_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}(a), \tag{16.2.2}$$

where $g^{(0)}_{\mu\nu} = dim(-\xi^2, 1, 1, 1)$, $g^{(1)}_{0i} = -\frac{ap^i}{2}$, $g^{(1)}_{11} = g^{(1)}_{22} = g^{(1)}_{33} = -ap^0$ and all other components are zero. Here, p^0, p^1, p^2 and p^3 are energy and momentum scales associated to the probe.

16.3 Deformed Equation of Motion and Energy-Momentum Tensor

We commence with the Lagrangian for scalar theory valid up to the first order in deformation parameter a in the κ-deformed Rindler space-time as (see [7])

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2\xi^2}(\partial_0\phi)^2 - \frac{1}{2}\Big((\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_\xi\phi)^2\Big) - \frac{ap^0}{2}\Big((\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_\xi\phi)^2\Big) \\ & + \frac{ap^1}{4\xi^2}\partial_0\phi\partial_x\phi + \frac{ap^2}{4\xi^2}\partial_0\phi\partial_y\phi + \frac{ap^3}{4\xi^2}\partial_0\phi\partial_\xi\phi - \frac{1}{2}V(x)\phi^2,\end{aligned} \tag{16.3.1}$$

where $V(x) = \lambda_1\delta(\xi - \xi_1) + \lambda_2\delta(\xi - \xi_2)$ and the last term represents the interaction of the plates with the field ϕ. Here λ_1 and λ_2 are coupling constants with dimension of length^{-1}. We find the corresponding action in κ-deformed Rindler space-time as

$$\mathcal{S}_{\kappa\text{-Rindler}} = \int d^4x\sqrt{-\hat{g}}\,\mathcal{L}, \tag{16.3.2}$$

where $\hat{g} = \det \hat{g}_{\mu\nu} = -\xi^2(1 - 3ap^0)$. The equation of motion following from the above action is

$$\Big\{\frac{1}{\xi^2}\partial_\tau^2 - \frac{1}{\xi}\partial_\xi(\xi\partial_\xi) - \nabla_\perp^2 + V(x) - ap^0\Big(\frac{1}{\xi}\partial_\xi(\xi\partial_\xi) + \nabla_\perp^2\Big) + \frac{a}{2\xi^2}\Big(p^1\partial_\tau\partial_x + p^2\partial_\tau\partial_y + p^3\partial_\tau\partial_\xi\Big)\Big\}\phi = 0, \tag{16.3.3}$$

where $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$. The symmetrized energy-momentum tensor ensuing from the above action is

$$\hat{T}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\Big(\hat{g}_{\mu\nu}(a) + \hat{g}_{\nu\mu}(a)\Big)\mathcal{L}. \tag{16.3.4}$$

16.4 Deformed Green's Function

In this Section, we solve the equation of motion given in Eq. (16.3.3) and find the Green's function in different regions of interest. Thus, using Eq. (16.3.3) we find that the corresponding Green's function obey

$$-\Big\{-\frac{1}{\xi^2}\partial_\tau^2 + \frac{1}{\xi}\partial_\xi(\xi\partial_\xi) + \nabla_\perp^2 - V(x) - \frac{a}{2\xi^2}\Big(p^1\partial_\tau\partial_x + p^2\partial_\tau\partial_y + p^3\partial_\tau\partial_\xi\Big) + ap^0\Big(\frac{1}{\xi}\partial_\xi(\xi\partial_\xi) + \nabla_\perp^2\Big)\Big\}\tilde{G}(x, x', a) = (1 + \frac{3}{2}ap^0)\frac{\delta(\tau - \tau')\delta(\xi - \xi')\delta(x_\perp - x'_\perp)}{\xi}. \tag{16.4.1}$$

Substituting $\tilde{G}(x, x', a) = \int_\infty^\infty \frac{d\omega d^2k_\perp}{8\pi^3} e^{-i\omega(\tau-\tau')}e^{i\mathbf{k}_\perp\cdot(\mathbf{x}_\perp-\mathbf{x}'_\perp)}\tilde{g}(\xi, \xi', a)$ in Eq. (16.4.1), we find the reduced Green's function as

$$\tilde{g}(\xi, \xi', a) = (1 + ap^0\frac{1}{2})\tilde{g}^{(0)}(\xi, \xi') + ap^0\Big[\int d\bar{\xi}\,\bar{\xi}\tilde{g}^{(0)}(\xi, \bar{\xi})\Big\{\frac{\zeta^2}{\bar{\xi}^2} + \lambda_1\delta(\bar{\xi} - \xi_1) + \lambda_2\delta(\bar{\xi} - \xi_2) - \frac{1}{p^0}\Big(\frac{p^1}{2\bar{\xi}^2}(\omega k_x) + \frac{p^2}{2\bar{\xi}^2}(\omega k_y) + \frac{p^3}{2\bar{\xi}^2}(-i\omega)\partial_{\bar{\xi}}\Big)\Big\}\tilde{g}^{(0)}(\bar{\xi}, \xi')\Big]. \tag{16.4.2}$$

Here we have, for $\{\xi, \xi'\} < \xi_1 < \xi_2$,

$$\tilde{g}^{(0)}(\xi, \xi') = I_< K_> - \frac{\lambda_1 \xi_1 K_1^2 + \lambda_2 \xi_2 K_2^2 - \lambda_1 \lambda_2 \xi_1 \xi_2 K_1 K_2 (K_2 I_1 - K_1 I_2)}{\tilde{\Delta}} I I_\prime. \tag{16.4.3}$$

For $\xi_1 < \{\xi, \xi'\} < \xi_2$,

$$\tilde{g}^{(0)}(\xi, \xi') = I_< K_> - \frac{\lambda_2 \xi_2 K_2^2 (1 + \lambda_1 \xi_1 K_1 I_1)}{\tilde{\Delta}} I I_\prime - \frac{\lambda \xi_1 I_1^2 (1 + \lambda_2 \xi_2 K_2 I_2)}{\tilde{\Delta}} K K_\prime$$
$$+ \frac{\lambda_1 \lambda_2 \xi_1 \xi_2 I_1^2 K_2^2}{\tilde{\Delta}} (I K_\prime + K I_\prime) \tag{16.4.4}$$

and for $\{\xi, \xi'\} > \xi_2 > \xi_1$,

$$\tilde{g}^{(0)}(\xi, \xi') = I_< K_> - \frac{\lambda_1 \xi_1 I_1^2 + \lambda_2 \xi_2 I_2^2 + \lambda_1 \lambda_2 \xi_1 \xi_2 I_1 I_2 (I_2 K_1 - I_1 K_2)}{\tilde{\Delta}} K K_\prime. \tag{16.4.5}$$

Here $\tilde{\Delta} = (1 + \lambda_1 \xi_1 K_1 I_1)(1 + \lambda_2 \xi_2 K_2 I_2) - \lambda_1 \lambda_2 \xi_1 \xi_2 I_1^2 K_2^2$ and we have used $I_i = I_\zeta(k_\perp \xi_i)$, $K_i = K_\zeta(k_\perp \xi_i)$; where $i = 1, 2$, $I = I_\zeta(k_\perp \xi)$, $I_\prime = I_\zeta(k_\perp \xi')$, $K = K_\zeta(k_\perp \xi)$ and $K_\prime = K_\zeta(k_\perp \xi')$. Note here that the range of the integration in the third term inside the square bracket in the Eq. (16.4.2) is either $\xi \to \xi'$ for $\xi' > \xi$ or $\xi' \to \xi$ for $\xi > \xi'$.

16.5 Force Acting on the Casimir Apparatus

We start with the general expression of the force density in curved space-time, $f^\nu = -\nabla_\mu T^{\mu\nu}$ [11] and using the expression of κ-deformed Rindler metric given in Eq. (16.2.2) we find force density as $\hat{f}_\xi = -(1 + ap^0)\left(\frac{1}{\xi}\hat{T}_{\xi\xi} + \partial_\xi(\tilde{T}_{\xi\xi})\right) - \frac{1}{\xi^3}\hat{T}_{00}$. Then using energy-momentum (see Eq. (16.3.4)), we find the total change in the momentum of the Casimir apparatus in ξ direction as $\Delta P_\xi = \int d\tau dx dy d\xi \sqrt{-\hat{g}}\, \langle \hat{f}_\xi \rangle$. Thus using $\langle \phi(x')\phi(x) \rangle = \frac{1}{i}\tilde{G}(x, x', a)$ and Eqs. (16.4.2)–(16.4.5), we find the total force, valid up to the first order in deformation parameter a to be $F_\xi = \frac{\Delta P_\xi}{\xi_0 \int d\tau dx dy}$. Taking the Minkowski-space limit, i.e., $\xi \to \infty$, (see [6]), we find the total force acting on parallel plates as

$$F_\xi = -(1 - ap^0)\frac{1}{96\pi^2 L^3 \xi_0}\Bigg[\int_0^\infty dY Y^2 \frac{1}{\left(1 + \frac{Y}{\lambda_1 L}\right)} + \int_0^\infty dY Y^2 \frac{1}{\left(1 + \frac{Y}{\lambda_2 L}\right)}$$
$$- \int_0^\infty dY Y^3 \frac{1 + \frac{1}{Y + \lambda_1 L} + \frac{1}{Y + \lambda_2 L}}{\left(1 + \frac{Y}{\lambda_1 L}\right)\left(1 + \frac{Y}{\lambda_2 L}\right) e^Y - 1}\Bigg] = -\frac{(\tilde{E}_{d1} + \tilde{E}_{d2} + \tilde{E}_c)}{\xi_0}. \tag{16.5.1}$$

For each plate, we get $\tilde{E}_{\text{total}} = \tilde{E}_{d1} + m_1 + \tilde{E}_{d2} + m_2 + \tilde{E}_c = \tilde{M}_1 + \tilde{M}_2 + \tilde{E}_c$, where $\tilde{M}_1$ and $\tilde{M}_2$ are interpretted as the renormalized masses for the plates and m_1 and m_2 are the bare mass of the plates. Thus, from Eq. (16.5.1) we find the gravitational force on the Casimir apparatus as

$$F_\xi = -(\tilde{M}_1 + \tilde{M}_2 + \tilde{E}_c)\frac{1}{\xi_0} = -(\tilde{M}_1 + \tilde{M}_2 + \tilde{E}_c)g. \tag{16.5.2}$$

Note that the negative sign in the above equation indicates the downward acceleration of gravity, and we have set $\frac{1}{\xi_0} = g$. This shows that the Casimir energy fall in accordance with the equivalence principle in κ-deformed space-time.

16.6 Conclusions

In this study, we investigated the influence of the non-commutativity of space-time on the fall of Casimir energy in the κ-deformed space-time. Our study shows that the κ-deformed Casimir energy is accelerated downward in the weak gravitational field. This implies that Casimir energy in κ-deformed space-time respects the equivalence principle.

Acknowledgements S.K.P thanks UGC, India, for the support through the JRF scheme (id.191620059604).

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