Electromagnetism ①

I Maxwell's equations

① $$\vec{\nabla} \cdot \vec{E}(\vec{r},t) = 4\pi\, \rho(\vec{r},t)$$

$$\vec{\nabla} \times \vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r},t) = 0$$

$$\vec{\nabla} \times \vec{B}(\vec{r},t) = \frac{1}{c}\frac{\partial \vec{E}(\vec{r},t)}{\partial t} + \frac{4\pi}{c}\vec{J}(\vec{r},t)$$

② Lorentz force on a charged particle

$$\vec{F}_a = q_a \vec{E}(\vec{r}_a,t) + q_a \frac{\vec{V}_a}{c} \times \vec{B}(\vec{r}_a,t)$$

③ Maxwell's eqns in vaccum ($\rho(\vec{r},t) = 0$, $\vec{J}(\vec{r},t) = 0$) is

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

II Electric and magnetic potential

1. $\vec{\nabla} \cdot \vec{B} = 0$

   A sufficient condition for the form of $\vec{B}$ is

   $$\vec{B} = \vec{\nabla} \times \vec{A} . \qquad (\because \ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0)$$

2. Is $\vec{A}(\vec{r},t)$ unique ? No.

3. Using this in the other source-less (homogeneous eqn.)

   we get

   $$\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = 0$$

   $$\vec{\nabla} \times \left[\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right] = 0$$

4. Again a sufficient requirement is

   $$\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = - \vec{\nabla} \phi \qquad (\vec{\nabla} \times (\vec{\nabla} \phi) = 0)$$

5. → The −ve sign in 4 is a conventional choice.

   → Is $\phi(\vec{r},t)$ unique ? No.

6. $$\vec{B} = \vec{\nabla} \times \vec{A}$$

   $$\vec{E} = - \vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

   $\vec{A}(\vec{r},t)$ — Magnetic vector potential

   $\phi(\vec{r},t)$ → Electric scalar potential

III Gauge transformation (non-uniqueness of ϕ and $\vec{A}$)

① → $\vec{E}$ and $\vec{B}$ are the physical quantities.

→ ϕ and $\vec{A}$ are mathematical constructs. (Aharanov-Bohm effect in 1959 claimed that $\vec{A}$ can be experimentally measured.)

② Let $\vec{A}' = \vec{A} + \vec{\nabla}\lambda(\vec{r},t)$

where $\lambda(\vec{r},t)$ is an arbitrary function.

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\left[\vec{A} + \vec{\nabla}\lambda\right]$$
$$= \vec{\nabla}\times\vec{A} = \vec{B}$$

③ Also

$$\vec{E}' = -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial\vec{A}'}{\partial t}$$
$$= -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial}{\partial t}\left[\vec{A} + \vec{\nabla}\lambda\right]$$
$$= -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\lambda$$

④ If we require $\vec{E}' = \vec{E}$ then we need

$$\phi' = \phi - \frac{1}{c}\frac{\partial\lambda(\vec{r},t)}{\partial t}$$

(5) Thus Maxwell's eqns when written in terms of ϕ and $\vec{A}$ are covariant under the transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

$$\phi' = \phi - \frac{1}{c}\frac{\partial \lambda}{\partial t}.$$

This is called gauge transformations.

(6) Notice that physical variables ($\vec{E}$ and $\vec{B}$) do not change under gauge transformation. Gauge dependence is an artifact of the mathematical construct involved in introducing ϕ and $\vec{A}$.

IV Maxwell's equations in terms of ϕ and $\vec{A}$

(1)

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

(2) Using these in the two Maxwell's equations involving the sources we have.

$$\vec{\nabla}\cdot\left[-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right] = 4\pi\rho$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{1}{c}\frac{\partial}{\partial t}\left[-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right] = \frac{4\pi}{c}\vec{J}$$

(3)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

(4) Thus we have.

$$-\nabla^2\phi - \frac{1}{c}\vec{\nabla}\cdot\frac{\partial \vec{A}}{\partial t} = 4\pi\rho$$

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\phi = \frac{4\pi}{c}\vec{J}$$

V Maxwell's equations in the Lorenz gauge

1. Let us choose a gauge in which

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 .$$

This is called the Lorentz gauge.

2. Using 1 the Maxwell's eqn. in 4 - IV becomes.

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = 4\pi\rho$$

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{J}$$

VI Delta function

① A particular representation of delta function is

$$\delta(x-a) = \underset{\epsilon\to 0}{Lt} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} e^{-\frac{(x-a)^2}{2\epsilon^2}} .$$

For practical purposes it is enough if we write

$$\delta(x-a) = \begin{cases} 0 & \text{if} \quad x \neq a \\ \infty & \text{if} \quad x = a. \end{cases}$$

②
$$\int_{-\infty}^{+\infty} dx\, \delta(x-a) = \int_{-\infty}^{+\infty} dx \underset{\epsilon\to 0}{Lt} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} e^{-\frac{(x-a)^2}{2\epsilon^2}}$$

$$= \underset{\epsilon\to 0}{Lt} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \int_{-\infty}^{+\infty} dx\, e^{-\frac{(x-a)^2}{2\epsilon^2}}$$

$$= \underset{\epsilon\to 0}{Lt} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \sqrt{\pi\, 2\epsilon^2}$$

$$= \underline{\underline{1}}$$

③ $$\int_{-\infty}^{+\infty} dx\, f(x)\, \delta(x-a) = \int_{-\infty}^{+\infty} dx\, f(x) \underset{\epsilon\to 0}{Lt} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} e^{-\frac{(x-a)^2}{2\epsilon^2}}$$

$$= \underset{\epsilon\to 0}{Lt} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \int_{-\infty}^{+\infty} dx\, e^{-\frac{(x-a)^2}{2\epsilon^2}} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= \underset{\epsilon\to 0}{Lt} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \int_{-\infty}^{+\infty} dx\, (x-a)^n\, e^{-\frac{(x-a)^2}{2\epsilon^2}}$$

$$\frac{(x-a)^2}{2\epsilon^2} = y \qquad (x-a) = \epsilon\sqrt{2}\sqrt{y} \qquad dx = \frac{\epsilon}{\sqrt{2}} \frac{dy}{\sqrt{y}}$$

$$= \underset{\epsilon\to 0}{Lt} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\, 2 \int_0^{\infty} \frac{\epsilon}{\sqrt{2}} \frac{dy}{\sqrt{y}} (\epsilon\sqrt{2}\sqrt{y})^n\, e^{-y}$$

$$= \underset{\epsilon\to 0}{Lt} \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon} \frac{f^{(n)}(a)}{n!} \sqrt{2}\,\epsilon\, (\sqrt{2}\,\epsilon)^n \int_0^{\infty} \frac{dy}{y}\, y^{\frac{n}{2}+1} e^{-y}$$

$$= \underset{\epsilon\to 0}{Lt} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \frac{1}{\sqrt{\pi}} (\sqrt{2}\,\epsilon)^n\, \Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= \frac{f^{(0)}(a)}{0!} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

$$= \underline{f(a)}$$

(4) Properties of delta function:

(i) $$\delta(x-a) = \begin{cases} 0 & \text{if} \quad x \neq a \\ \infty & \text{if} \quad x = a \end{cases}$$

(ii) $$\int_{-\infty}^{+\infty} dx\, \delta(x-a) = 1$$

(iii) $$\int_{-\infty}^{+\infty} dx\, \delta(x-a)\, f(x) = f(a)$$