Electromagnetism ②

I. Statement of conservation of charge

① $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$ $\qquad$ $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

② $\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} = \frac{4\pi}{c}\frac{\partial\rho}{\partial t}$

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=0} - \frac{1}{c}\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J}$$

$$\Rightarrow \quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

③ Integrating ② over a volume bounded by surface S

$$\int_V d^3r\, \frac{\partial}{\partial t}\rho(\vec{r},t) + \int_V d^3r\, \vec{\nabla}\cdot\vec{J}(\vec{r},t) = 0$$

$$\frac{\partial}{\partial t}\underbrace{\int_V d^3r\, \rho(\vec{r},t)}_{Q(t)} + \oint d\vec{s}\cdot\vec{J}(\vec{r},t) = 0$$

$$\frac{\partial}{\partial t}Q(t) + \oint d\vec{s}\cdot\vec{J}(\vec{r},t) = 0$$

II Maxwell's equations for statics

① Let ρ, $\vec{J}$, $\vec{E}$ & $\vec{B}$ be time independent.

② $\vec{\nabla} \cdot \vec{E}(\vec{r}) = 4\pi \rho(\vec{r})$

$\vec{\nabla} \times \vec{E}(\vec{r}) + \frac{1}{c}\frac{\partial}{\partial t}\vec{B}(\vec{r}) = 0$

$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$

$\vec{\nabla} \times \vec{B}(\vec{r}) - \frac{1}{c}\frac{\partial \vec{E}(\vec{r})}{\partial t} = \frac{4\pi}{c}\vec{J}(\vec{r})$

③ $\vec{\nabla} \cdot \vec{E}(\vec{r}) = 4\pi \rho(\vec{r})$

$\vec{\nabla} \times \vec{E}(\vec{r}) = 0$

$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$

$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{4\pi}{c}\vec{J}(\vec{r})$

Thus the $\vec{E}$ and $\vec{B}$ fields are de-coupled.
Thus we have electrostatics and magnetostatics

④ Show that statement of conservation of charge in the static case is $\vec{\nabla} \cdot \vec{J} = 0$. Show that this is consistant with $\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{4\pi}{c}\vec{J}(\vec{r})$.

III/ Electrostatics

① $\vec{\nabla} \cdot \vec{E}(\vec{r}) = 4\pi \rho(\vec{r})$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = 0$$

② $\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$

$$-\nabla^2 \phi(\vec{r}) = 4\pi \rho(\vec{r}) \quad \leftarrow \text{Laplace equation}$$

IV Green's function : potential due to a unit point charge

① Charge density of a point charge of charge unity is situated at the point $\vec{r}'$ is

$$\rho(\vec{r}) = \delta^{(3)}(\vec{r} - \vec{r}')$$

② Let us denote the potential due to this point charge as $G(\vec{r}, \vec{r}')$. Using III -② we have.

$$-\nabla^2 G(\vec{r}, \vec{r}') = 4\pi\, \delta^{(3)}(\vec{r} - \vec{r}')$$

V Evaluation of potential due to a point charge.

① $$-\nabla^2 G(\vec{r},\vec{r}\,') = 4\pi\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

② Observe that $G(\vec{r},\vec{r}\,') = G(\vec{r}-\vec{r}\,')$. $\longrightarrow$ Translation!

③ Fourier transformation

$$f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \tilde{f}(k)\, e^{ikx} \qquad \delta(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{ikx}$$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} dx\, f(x)\, e^{-ikx} \qquad 1 = \int_{-\infty}^{+\infty} dx\, \delta(x)\, e^{-ikx}$$

④ $$-\nabla^2 \int_{-\infty}^{+\infty} \frac{d^3k}{(2\pi)^3}\, \tilde{G}(\vec{k})\, e^{i\vec{k}.(\vec{r}-\vec{r}\,')} = 4\pi \int_{-\infty}^{+\infty} \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}.(\vec{r}-\vec{r}\,')}$$

$$\int_{-\infty}^{+\infty} \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}.(\vec{r}-\vec{r}\,')} \left[k^2 \tilde{G}(\vec{k}) - 4\pi \right] = 0$$

$$\Rightarrow \quad \tilde{G}(\vec{k}) = \frac{4\pi}{k^2}$$

⑤ $$G(\vec{r},\vec{r}\,') = \int_{-\infty}^{+\infty} \frac{d^3k}{(2\pi)^3}\, \tilde{G}(\vec{k})\, e^{i\vec{k}.(\vec{r}-\vec{r}\,')} \qquad R \equiv |\vec{r}-\vec{r}\,'|$$

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty k^2 dk \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, \frac{1}{k^2}\, e^{ikR\cos\theta}$$

choosing $\vec{R}$ along the z-axis.

(6) $$G(\vec{r}, \vec{r}') = \frac{1}{\pi} \int_0^\infty dk \int_0^\pi \sin\theta \, d\theta \; e^{ikR\cos\theta}$$

$$ikR\cos\theta = t \qquad ikR\sin\theta\, d\theta = -dt$$

$$= \frac{1}{\pi} \int_0^\infty dk \, (-1) \int_{ikR}^{-ikR} \frac{dt}{ikR} \; e^{t}$$

$$= \frac{1}{\pi} \int_0^\infty dk \; \frac{e^{ikR} - e^{-ikR}}{ikR}$$

$$= \frac{2}{\pi} \int_0^\infty dk \; \frac{\sin kR}{kR} \qquad\qquad kR = x, \quad R\,dk = dx.$$

$$= \frac{1}{R} \; \frac{2}{\pi} \int_0^\infty dx \; \frac{\sin x}{x}.$$

$$= \frac{1}{R} \qquad \left(\text{using} \quad \int_0^\infty dx \; \frac{\sin x}{x} = \frac{\pi}{2}\right)$$

$$= \frac{1}{|\vec{r} - \vec{r}'|}$$

VI Point charge

① Potential due to a point charge 'q'

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}'|}$$

② Electric field due to a point charge 'q'

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

$$= q\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

③ Force on a charge q_a at position $\vec{r}_a$ due to a point charge 'q'

$$\vec{F} = q_a\,\vec{E}(\vec{r}_a)$$

$$= q_a\,q\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

④ Energy of a configuration of point charges

$$U = \sum_i \sum_{\substack{j \\ i\neq j}} \frac{q_i\,q_j}{|\vec{r}_i-\vec{r}_j|}$$