Electromagnetism (3)

I Electric potential and electric fields

(1) $$-\nabla^2 \phi(\vec{r}) = 4\pi\, \rho(\vec{r})$$

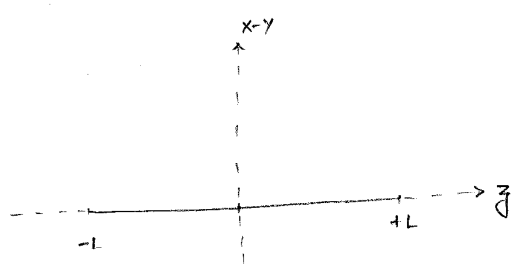
$$-\nabla^2 G(\vec{r}, \vec{r}') = 4\pi\, \delta^{(3)}(\vec{r}-\vec{r}')$$

$$\phi(\vec{r}) = \int d^3r'\, G(\vec{r}, \vec{r}')\, \rho(\vec{r}')$$

where $$G(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$$

II (1) Electric potential due to a line segment of length $2L$ with a uniform charge density 'λ'.

(2) We shall calculate the electric potential at a distance 'r' above the midpoint of the rod.



(3) $$\rho(\vec{r}) = \lambda\, \delta(x)\, \delta(y)\, f(z)$$

where $$f(z) = \begin{cases} 0 & \text{if } z > L \ \& \ z < -L \\ 1 & \text{if } -L < z < +L \end{cases}$$

(4) $$\phi(\vec{r}) = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \frac{\lambda\, \delta(x')\, \delta(y')\, f(z')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \int_{-L}^{+L} dz' \frac{\lambda}{\sqrt{x^2+y^2+(z-z')^2}}$$

$x^2+y^2 \equiv R^2.$

$z'-z = t \qquad dz' = dt$

$$= \int_{-L-z}^{L-z} dt \frac{\lambda}{\sqrt{R^2+t^2}}$$

$t = R \sinh\theta$

$dt = R \cosh\theta\, d\theta$

$$= \int_{-\sinh^{-1}\left(\frac{L+z}{R}\right)}^{\sinh^{-1}\left(\frac{L-z}{R}\right)} R\cosh\theta\, d\theta \frac{\lambda}{R\cosh\theta}$$

$$= \lambda \left[\sinh^{-1}\left(\frac{L-z}{R}\right) + \sinh^{-1}\left(\frac{L+z}{R}\right) \right]$$

(5) $$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

(6) $$E_x(\vec{r}) = -\frac{\partial}{\partial x}\phi(\vec{r}) = -\lambda \frac{\partial}{\partial x}\left[\sinh^{-1}\left(\frac{L-z}{\sqrt{x^2+y^2}}\right) + \sinh^{-1}\left(\frac{L+z}{\sqrt{x^2+y^2}}\right) \right]$$

$$= -\lambda \frac{1}{\sqrt{1+\frac{(L-z)^2}{(x^2+y^2)}}} \left(-\frac{1}{2}\right) \frac{2x(L-z)}{(x^2+y^2)^{3/2}} - \lambda \frac{1}{\sqrt{1+\frac{(L+z)^2}{(x^2+y^2)}}} \left(-\frac{1}{2}\right) \frac{2x(L+z)}{(x^2+y^2)^{3/2}}.$$

$$= \frac{\lambda x (L-z)}{(x^2+y^2)\sqrt{x^2+y^2+(L-z)^2}} + \frac{\lambda x (L+z)}{(x^2+y^2)\sqrt{x^2+y^2+(L+z)^2}}$$

(7) $$E_y(\vec{r}) = \frac{\lambda y (L-z)}{(x^2+y^2)\sqrt{x^2+y^2+(L-z)^2}} + \frac{\lambda y (L+z)}{(x^2+y^2)\sqrt{x^2+y^2+(L+z)^2}}$$

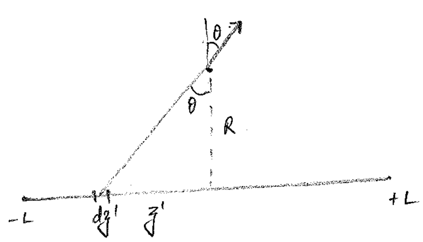
(8) $$E_z(\vec{r}) = \frac{\lambda}{\sqrt{x^2+y^2+(L-z)^2}} - \frac{\lambda}{\sqrt{x^2+y^2+(L+z)^2}}$$

(9) Thus we have

$$\vec{E}(x,y,z=0) = \frac{2\lambda L \hat{R}}{R^2\sqrt{R^2+L^2}}$$

Direct calculation of $\vec{E}(x,y,z=0)$

(1)



$dq = \lambda dz'$

(2) $$|d\vec{E}| = \frac{dq}{R^2+z'^2}$$

(3) $$\vec{E}(x,y,z=0) = \int_0^L \frac{2\, dq \cos\theta}{R^2+z'^2}$$

$$= 2\lambda \int_0^L dz' \frac{1}{R^2+z'^2} \frac{R}{\sqrt{R^2+z'^2}}$$

(4) $\vec{E}(x,y,z=0) = 2\lambda R \int_0^L \frac{dz'}{(R^2+z'^2)^{3/2}}$

$$z' = R\tan\theta$$
$$dz' = R\sec^2\theta\, d\theta$$

$1+\tan^2\theta = \sec^2\theta$

$$= 2\lambda R \int_0^{\tan^{-1}\frac{L}{R}} \frac{R\sec^2\theta\, d\theta}{(R^2\sec^2\theta)^{3/2}}$$

$$= \frac{2\lambda}{R} \int_0^{\tan^{-1}\frac{L}{R}} \cos\theta\, d\theta$$

$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$

$$= \frac{2\lambda}{R} \sin\theta \Big|_{\theta=0}^{\theta=\tan^{-1}\frac{L}{R}}$$

$$= \frac{2\lambda}{R} \frac{\left(\frac{L}{R}\right)}{\sqrt{1+\left(\frac{L}{R}\right)^2}}$$

$$= \frac{2\lambda L}{R\sqrt{R^2+L^2}}$$

III Surface charge

1. Consider a disc of uniform charge density σ and radius 'a'.

2. $$\rho(\vec{r}) = \sigma\,\delta(z)\,f(x,y) \qquad f(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 < a^2 \\ 0 & \text{if } x^2+y^2 > a^2 \end{cases}$$

3. $$\phi(\vec{r}) = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \; \frac{\sigma\,\delta(z')\,f(x',y')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

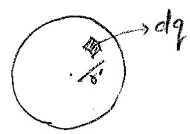
4. $$\phi(x=0,y=0,z) = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \; \frac{\sigma\, f(x',y')}{\sqrt{x'^2+y'^2+z^2}}$$

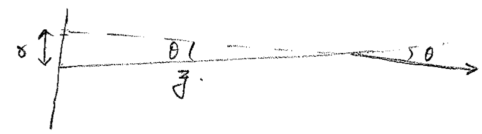
$$= \int_0^{2\pi} d\theta' \int_0^{a} r'dr \; \frac{\sigma}{\sqrt{r'^2+z^2}} \qquad r'^2+z^2 = y, \quad 2r'dr' = dy$$

$$= 2\pi\sigma \int_{z^2}^{a^2+z^2} \frac{dy}{2}\,\frac{1}{\sqrt{y}}$$

$$= 2\pi\sigma\left[\sqrt{a^2+z^2} - z\right]$$

5 Electric field due to a disc





$$dq = \sigma\, r' dr' d\theta'$$

$$|d\vec{E}| \cos\theta = \frac{dq}{r'^2 + z^2} \cos\theta$$

$$= \frac{dq}{r'^2 + z^2} \frac{z}{\sqrt{r'^2 + z^2}}$$

$$\vec{E}(x=0, y=0, z) = \int_0^{2\pi} \int_0^{a} \frac{\sigma\, r' dr' d\theta'\; z}{(r'^2 + z^2)^{3/2}}$$

$$= 2\pi \sigma z \int_0^a \frac{r' dr'}{(r'^2 + z^2)^{3/2}}$$

$$r'^2 + z^2 = t$$
$$2 r' dr' = dt$$

$$= 2\pi \sigma z \int_{z^2}^{a^2 + z^2} \frac{dt}{2} \frac{1}{t^{3/2}}$$

$$= 2\pi \sigma z \left\{ -\frac{1}{\sqrt{t}} \right\}_{t = z^2}^{t = a^2 + z^2}$$

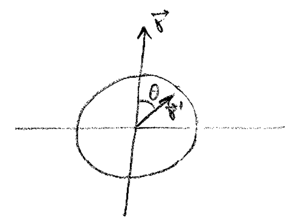
$$= 2\pi \sigma z \left[\frac{1}{z} - \frac{1}{\sqrt{a^2 + z^2}} \right]$$

$$= 2\pi \sigma \left[1 - \frac{z}{\sqrt{a^2 + z^2}} \right]$$

IV Volume charge

① Electric potential due to a sphere of radius 'a' and charge density ρ.

$|\vec{r}| > a$



② $$\phi(\vec{r}) = \int_{|\vec{r}'|<a} d^3r' \frac{\rho}{|\vec{r}-\vec{r}'|}$$

$$= \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \int_0^a r'^2 dr \frac{\rho}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}}$$

$\cos\theta = t$
$\sin\theta \, d\theta = -dt$

$$= 2\pi\rho \int_{-1}^{+1} dt \int_0^a r'^2 dr' \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}$$

$$r^2 + r'^2 - 2rr't = y \qquad -2rr'dt = dy$$

$$= -2\pi\rho \int_0^a r'^2 dr' \int_{(r-r')^2}^{(r+r')^2} \frac{dy}{-2rr'} \frac{1}{\sqrt{y}}$$

$$= -2\pi\rho \int_0^a r'^2 dr' \frac{-1}{rr'} \left\{ \sqrt{y} \right\}_{y=(r-r')^2}^{y=(r+r')^2}$$

$$= -2\pi\rho \int_0^a \frac{r'}{r} dr' \left[(r-r') - (r+r') \right]$$

$$= +\frac{4\pi\rho}{r} \int_0^a r'^2 dr'$$

$$= +\frac{4\pi\rho}{r} \frac{a^3}{3}$$

$$= +\frac{\rho}{r} \frac{4\pi}{3} a^3$$

③ $|\vec{r}| < a$

$$\phi(\vec{r}) = -2\pi\rho \int_0^a r'^2 dr' \int_{(r-r')^2}^{(r+r')^2} \frac{dy}{-2rr'} \frac{1}{\sqrt{y}}$$

$$= -2\pi\rho \int_0^r r'^2 dr' \int_{(r-r')^2}^{(r+r')^2} \frac{dy}{-2rr'} \frac{1}{\sqrt{y}} \quad + 2\pi\rho \int_r^a r'^2 dr' \int_{(r-r')^2}^{(r+r')^2} \frac{dy}{-2rr'} \frac{1}{\sqrt{y}}$$

$r' < r < a$ $\qquad$ $r < r' < a$

$$= -2\pi\rho \int_0^r \frac{r'}{r} dr' (-1) \left\{\sqrt{y}\right\}_{y=(r-r')^2}^{y=(r+r')^2} \quad + 2\pi\rho \int_r^a \frac{r'}{r} dr' (-1) \left\{\sqrt{y}\right\}_{y=(r-r')^2}^{y=(r+r')^2}$$

$$= -\frac{2\pi\rho}{r} \int_0^r r' dr' \left[(r-r') - (r+r') \right] \quad + \frac{2\pi\rho}{r} \int_r^a r' dr' \left[(r'-r) - (r+r') \right]$$

$(\because\ r' < r)$ $\qquad$ $(\because\ r' > r)$

$$= + \frac{4\pi\rho}{r} \int_0^r r'^2 dr' \quad + 4\pi\rho \int_r^a r' dr'$$

$$= + \frac{4\pi\rho}{r} \frac{r^3}{3} \quad + 4\pi\rho \left[\frac{a^2}{2} - \frac{r^2}{2} \right]$$

$$= -\frac{4\pi\rho}{6} r^2 + 2\pi\rho a^2.$$

④ $$\phi(\vec{r}) = \begin{cases} + \dfrac{\rho}{r} \dfrac{4\pi}{3} a^3 & r > a. \\ -\dfrac{4\pi\rho}{6} r^2 + 2\pi\rho a^2 & r < a \end{cases}$$

5 $$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

$$= -\frac{\partial}{\partial r}\phi(r)$$

6 $r > a$ $$\vec{E}(\vec{r}) = -\frac{\partial}{\partial r}\left((+1)\frac{\rho}{r}\frac{4\pi}{3}a^3\right)$$

$$= -\rho\frac{4\pi}{3}a^3\frac{(-1)}{r^2}.$$

$$= \frac{\rho}{r^2}\frac{4\pi}{3}a^3$$

7 $r < a$ $$\vec{E}(\vec{r}) = -\frac{\partial}{\partial r}\left\{-\frac{4\pi\rho}{6}r^2 + 2\pi\rho a^2\right\}$$

$$= \frac{4\pi\rho}{6}2r + 0$$

$$= \frac{4\pi}{3}\rho r = \frac{\rho}{r^2}\frac{4\pi}{3}r^3$$

8 $$\vec{E}(\vec{r}) = \begin{cases} \frac{\rho}{r^2}\frac{4\pi}{3}a^3 & r > a \\ \frac{\rho}{r^2}\frac{4\pi}{3}r^3 & r < a \end{cases}$$