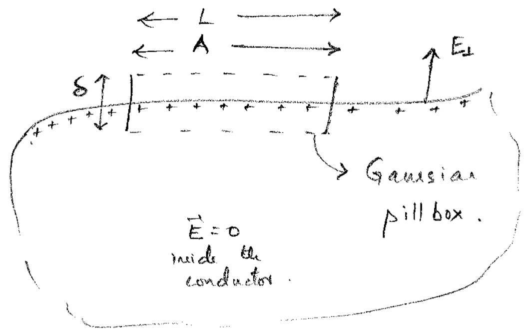
Electromagnetism 6

I. Boundary conditions on the surface of a conductor



① $$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{E} = 0$$

② $$\oint d\vec{a}\cdot\vec{E} = 4\pi\int da\,\sigma$$

σ → surface charge density

$$E_\perp A + E_\parallel \delta = 4\pi\sigma A$$

③ $$\oint d\vec{l}\cdot\vec{E} = 0$$

$$E_\parallel L + E_\perp(x)\frac{\delta}{2} - E_\perp(x+dx)\frac{\delta}{2} = 0$$

$$E_\parallel L + O(dx\,.\,\delta) = 0$$

$$\Rightarrow \quad E_\parallel L = 0$$

$$\Rightarrow \quad E_\parallel = 0$$

④ Using ③ in ②

$$E_\perp = 4\pi\sigma$$

II) Boundary value problems

① Conductors fix the potential $\phi(\vec{r})$ on its surface. Thus we get a class of problems which involves solving

$$-\nabla^2 \phi(\vec{r}) = 4\pi \rho(\vec{r})$$

for a prescribed conducting surface which might either be insulated (fixed charge) or maintained at a fixed potential (grounded means $\phi = 0$).

② Once we have solved for $\phi(\vec{r})$ we have

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

③ Using I we have the charge density on the conducting surface to be

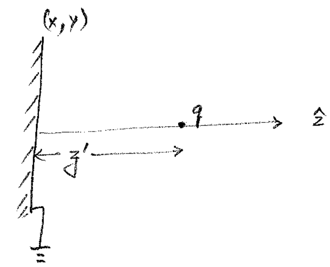
$$\sigma(\vec{r}_s)\,|d\vec{a}| = \frac{1}{4\pi}\,\vec{E}(\vec{r}_s)\cdot d\vec{a}$$

$\vec{r}_s \equiv$ point on the conducting surface.

④ Force on the conducting surface is

$$\vec{F} = \int_{\text{surface}} |d\vec{a}|\;\sigma(\vec{r}_s)\;\vec{E}(\vec{r}_s)$$

III Method of images — infinite conducting plane

1. 

2. $$-\nabla^2 \phi(\vec{r}) = q\, \delta(x)\, \delta(y)\, \delta(z - z')$$

with the boundary condition $\phi(x, y, z=0) = 0$.

3. The above problem is equivalent to

$$-\nabla^2 \phi(\vec{r}) = q\, \delta(x)\, \delta(y)\, \delta(z-z') - q\, \delta(x)\, \delta(y)\, \delta(z+z')$$

with the usual vanishing boundary condit at infinity.

4. We know the solution to 3 is

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - z'\hat{z}|} - \frac{q}{|\vec{r} + z'\hat{z}|}$$

5. $$\phi(x, y, z=0) = \frac{q}{\sqrt{x^2 + y^2 + (\underbrace{z}_{=0}-z')^2}} - \frac{q}{\sqrt{x^2 + y^2 + (\underbrace{z}_{=0}+z')^2}} = 0$$

and thus the requirement in 2 is satisfied which demonstrates the equivalance.

(6) $$\vec{E}(\vec{r}) = - \vec{\nabla} \phi(\vec{r})$$

(7) $$\vec{E}(\vec{r})\Big|_{\substack{\text{on the}\\ \text{conductor}}} = - \left\{ \frac{\partial}{\partial z} \phi(x,y,z) \right\}_{z=0} \hat{k}$$

$$= - \left[\frac{\partial}{\partial z} \left\{ \frac{q}{\sqrt{x^2+y^2+(z-z')^2}} - \frac{q}{\sqrt{x^2+y^2+(z+z')^2}} \right\} \right]_{z=0} \hat{k}$$

$$= \left[\frac{q(z-z')}{\left\{x^2+y^2+(z-z')^2\right\}^{3/2}} - \frac{q(z+z')}{\left\{x^2+y^2+(z+z')^2\right\}^{3/2}} \right]_{z=0} \hat{k}$$

$$= - \frac{2qz'}{\left\{x^2+y^2+z'^2\right\}^{3/2}} \hat{k}$$

(8) $$\sigma(x,y,z=0) = - \frac{1}{4\pi} \frac{2qz'}{\left\{x^2+y^2+z'^2\right\}^{3/2}}$$

charge density on conductor

(9) $$\vec{F} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \sigma(x,y,z=0) \; \vec{E}(x,y,z=0)$$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \frac{(-1)}{4\pi} \frac{2qz'}{(x^2+y^2+z'^2)^{3/2}} \; (-1) \frac{2qz'}{(x^2+y^2+z'^2)^{3/2}} \hat{k}$$

$$= \frac{4q^2z'^2}{4\pi} \hat{k} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \frac{1}{(x^2+y^2+z'^2)^3}$$

(10) $$\vec{F} = \frac{4q^2 z'^2}{4\pi}\,\hat{k}\int_0^{2\pi} d\theta \int_0^{\infty} r\,dr\,\frac{1}{(r^2+z'^2)^3}$$

$r^2 + z'^2 = t$
$2r\,dr = dt$

$$= \frac{\cancel{4}q^2 z'^2}{\cancel{4\pi}}\,\hat{k}\;\cancel{2\pi}\int_{z'^2}^{\infty}\frac{dt}{\cancel{2}}\,\frac{1}{t^3}$$

$$= q^2 z'^2\,\hat{k}\left\{\frac{-1}{2t^2}\right\}_{z'^2}^{\infty}$$

$$= q^2 z'^2\,\hat{k}\,\frac{1}{2z'^4}$$

$$= \frac{q^2}{2z'^2}\,\hat{k}$$

← check factor of 2. should it not be $\frac{q^2}{4z'^2}$.