Electromagnetism ⑦-Ⓐ

I/ Magnetostatics

① $\vec{\nabla}\cdot\vec{E} = 4\pi\,\rho(\vec{r},t)$ $\qquad \vec{\nabla}\cdot\vec{B} = 0$

$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}(\vec{r},t)$

② Time independence

$\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0$ $\qquad$ also $\vec{\nabla}\cdot\vec{J} = 0$

$\vec{\nabla}\times\vec{B}(\vec{r}) = \frac{4\pi}{c}\vec{J}(\vec{r})$

③ $\vec{B} = \vec{\nabla}\times\vec{A}$

$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{J}(\vec{r})$

$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}(\vec{r}) = \frac{4\pi}{c}\vec{J}(\vec{r})$

④ Choose the gauge $\vec{\nabla}\cdot\vec{A} = 0$

$$-\nabla^2\vec{A}(\vec{r}) = \frac{4\pi}{c}\vec{J}(\vec{r})$$

$$\Rightarrow \quad \vec{A}(\vec{r}) = \frac{1}{c}\int d^3r'\,\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

⑤ Validity of gauge choice:

If $\vec{\nabla}\cdot\vec{A} \neq 0$

choose $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$

then $\vec{\nabla}\cdot\vec{A}' = 0$ requires

$$-\nabla^2\lambda = \vec{\nabla}\cdot\vec{A}$$

II Biot - Savart law

① $$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

② $$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$$

$$= \frac{1}{c}\int d^3r' \; \vec{J}(\vec{r}') \quad \vec{\nabla}\times\frac{1}{|\vec{r}-\vec{r}'|}$$

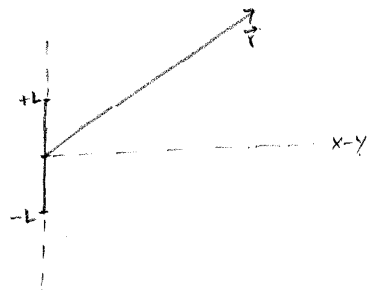
③ $$\vec{\nabla}\times\frac{1}{|\vec{r}-\vec{r}'|} = \epsilon_{ijk}\,\partial_j\,\frac{1}{\sqrt{(x-x')_m(x-x')_m}}$$

$$= -\,\epsilon_{ijk}\,\frac{x_j - x_j'}{|\vec{r}-\vec{r}'|^3}$$

④ $$\vec{B}(\vec{r}) = \frac{1}{c}\int d^3r' \quad \frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

III Magnetic field due to a wire segment

1.



2. $$\vec{J}(\vec{r}) = I\, \delta(x)\, \delta(y)\, \hat{k} \qquad -L < z < +L$$

$$r^2 = x^2 + y^2$$

3. $$\vec{A}(\vec{r}) = \frac{1}{c} \int d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{c} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-L}^{+L} dz' \frac{I\, \hat{k}\, \delta(x')\, \delta(y')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{I}{c}\, \hat{k} \int_{-L}^{+L} dz' \frac{1}{\sqrt{r^2 + (z-z')^2}} \qquad (A_x = 0,\ A_y = 0,\ A_z)$$

3. $$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$$

$$= \frac{I}{c}\, \hat{i} \int_{-L}^{+L} dz' \frac{\partial}{\partial y} \frac{1}{\sqrt{x^2 + y^2 + (z-z')^2}} - \frac{I}{c}\, \hat{j} \int_{-L}^{+L} dz' \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2 + (z-z')^2}}$$

$$= -\frac{I}{c} (y\, \hat{i} - x\, \hat{j}) \int_{-L}^{+L} dz' \frac{1}{[r^2 + (z-z')^2]^{3/2}}$$

(4)
$$\int_{-L}^{+L} dz' \frac{1}{\left[r^2 + (z-z')^2\right]^{3/2}} = \int_{z-L}^{z+L} dt \frac{1}{(r^2+t^2)^{3/2}}$$

$z - z' = t$

$-dz' = dt$

$t = r\tan\theta$

$dt = r\sec^2\theta\, d\theta$

$$= \int_{\tan^{-1}\frac{z-L}{r}}^{\tan^{-1}\frac{z+L}{r}} \frac{r \sec^2\theta\, d\theta}{r^3 \sec^3\theta}$$

$$= \frac{1}{r^2} \sin\theta \Big|_{\tan^{-1}\frac{z-L}{r}}^{\tan^{-1}\frac{z+L}{r}}$$

$$= \frac{1}{r^2}\left[\frac{\frac{z+L}{r}}{\sqrt{1+\left(\frac{z+L}{r}\right)^2}} - \frac{\frac{z-L}{r}}{\sqrt{1+\left(\frac{z-1}{r}\right)^2}} \right]$$

$$= \frac{1}{r^2}\left[\frac{z+L}{\sqrt{r^2+(z+L)^2}} - \frac{z-L}{\sqrt{r^2+(z-L)^2}} \right]$$

(5)
$$\hat{\phi} = \hat{z} \times \left[\hat{r} - \hat{r}\cdot\hat{z} \right] = -(y\hat{i} - x\hat{j})\frac{1}{r}$$

(6)
$$\vec{B}(\vec{r}) = \hat{\phi} \frac{I}{c} \frac{1}{r} \left[\frac{z+L}{\sqrt{r^2+(z+L)^2}} - \frac{z-L}{\sqrt{r^2+(z-L)^2}} \right]$$