Date: Aug 18, 2004

Dear Daniel,

I made inexcusable number of mistakes during our discussion today. I list the mistakes here and attach the correct solution.

Mistakes:

① $\hat{\phi}' = -\cos\phi'\,\hat{i} + \sin\phi'\,\hat{j}$

This does not come out of the $d\phi'$ integral.

② $|\vec{r}-\vec{r}'|^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi') + (z-z')^2$

③ $\vec{A}(\rho=0,\phi,z) = 0$.

This does not imply $\vec{B}(\rho=0,\phi,z) = 0$.

④ $\vec{J}(\vec{r}') = I\,n\,dz''\,\delta(\rho'-a)\,\delta(z'-z'')\,\hat{\phi}'$

This solves our concern as to how 'n' seeps in.

Kindly go through the attached solution. If possible work it out independently. $\vec{A}(\vec{r})$ for arbitrary $\vec{r}$ is not attempted because it leads to elliptic integrals.

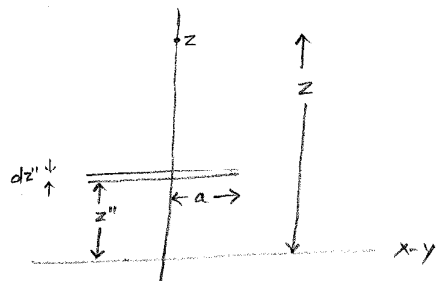
Will regards,

Shyam

Problem 5.11, Griffiths

Magnetic field due to a finite solenoid on the axis

(1)



(2) $\vec{J}(\vec{r}') = I_{dz''} \, \delta(\rho' - a)\, \delta(z' - z'')\, \hat{\phi}'$

where $I_{dz''} \equiv$ current passing through the infinitisimal circular wire of height dz''.

$= I \, n \, dz''$

where $I \equiv$ current passing through the coil

$n \equiv \dfrac{\text{no. of turns}}{\text{length}}$

(3) Cross check :

(i) Total current in the finite solenoid if viewed as a cylinder $\Big\} = I \, n \, (z_2 - z_1)$

(ii) check dimensions.

④ Thus we have

$$\vec{J}(\vec{r}') = I n \, dz'' \, \delta(\rho' - a) \, \delta(z' - z'') \, \hat{\phi}'$$

$$= I n \, dz'' \, \delta(\rho' - a) \, \delta(z' - z'') \left[- \cos\phi' \, \hat{i} + \sin\phi' \, \hat{j} \right]$$

⑤ $$|\vec{r} - \vec{r}'|^2 = (\rho \cos\phi - \rho' \cos\phi')^2 + (\rho \sin\phi - \rho' \sin\phi')^2 + (z - z')^2$$

$$= \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z - z')^2$$

⑥ $$\vec{A}_{dz''}(\rho, \phi, z) = \frac{1}{c} \int d^3 r' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{c} \int_0^{2\pi} a \, d\phi' \int_0^{\infty} d\rho' \int_{-\infty}^{+\infty} dz' \; \frac{I n \, dz'' \, \delta(\rho' - a) \, \delta(z' - z'') \left[- \cos\phi' \, \hat{i} + \sin\phi' \, \hat{j} \right]}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + (z - z')^2}}$$

$$= \frac{1}{c} \, I n \, dz'' \; a \int_0^{2\pi} d\phi' \; \frac{\left[- \cos\phi' \, \hat{i} + \sin\phi' \, \hat{j} \right]}{\sqrt{\rho^2 + a^2 - 2\rho a \cos(\phi - \phi') + (z - z')^2}}$$

⑦ $$\vec{A}_{dz''}(\rho = 0, \phi, z) = \frac{1}{c} \, I n \, dz'' \; \frac{a}{\sqrt{a^2 + (z - z'')^2}} \int_0^{2\pi} d\phi' \left[- \cos\phi' \, \hat{i} + \sin\phi' \, \hat{j} \right]$$

$$= \underline{\underline{0}}$$

⑧ CAUTION:

$$\vec{A}_{dz''}(\rho=0,\phi,z) = 0 \quad \not\Rightarrow \quad \vec{B}_{dz''}(\rho=0,\phi,z) = 0$$

⑨
$$\vec{B}_{dz''}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{B}_{dz''}(\rho=0,\phi,z) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')_{\rho=0}}{|\vec{r}-\vec{r}'|^3_{\rho=0}}$$

⑩
$$\vec{J}(\vec{r}') = I\, n\, dz''\, \delta(\rho'-a)\, \delta(z'-z'') \left[-\cos\phi'\, \hat{i} + \sin\phi'\, \hat{j}\right]$$

$$(\vec{r}-\vec{r}')_{\rho=0} = \left\{(\rho\cos\phi - \rho'\cos\phi')\,\hat{i} + (\rho\sin\phi - \rho'\sin\phi')\,\hat{j} + (z-z')\,\hat{k}\right\}_{\rho=0}$$

$$= -\rho'\cos\phi'\,\hat{i} - \rho'\sin\phi'\,\hat{j} + (z-z')\,\hat{k}$$

$$\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')_{\rho=0} = I n\, dz''\, \delta(\rho'-a)\, \delta(z'-z'') \left[(z-z')\cos\phi'\,\hat{i} + (z-z')\sin\phi'\,\hat{j} + \rho'\,\hat{k}\right]$$

⑪
$$\vec{B}_{dz''}(\rho=0,\phi,z)$$

$$= \frac{1}{c}\int_0^{2\pi} a\,d\phi' \int_0^{\infty} d\rho' \int_{-\infty}^{+\infty} dz' \, \frac{I n\, dz''\, \delta(\rho'-a)\, \delta(z'-z'') \left[(z-z')\cos\phi'\,\hat{i} + (z-z')\sin\phi'\,\hat{j} + \rho'\,\hat{k}\right]}{\left[\rho'^2 + (z-z')^2\right]^{3/2}}$$

(12) $$\vec{B}_{dz''}(\rho=0,\phi,z) = \frac{1}{c}\frac{I n\, a\, dz''}{\left[a^2+(z-z'')^2\right]^{3/2}}\int_0^{2\pi} d\phi'\left[(z-z'')\cos\phi'\,\hat{i} + (z-z'')\sin\phi'\,\hat{j} + a\,\hat{k}\right]$$

$$= \frac{1}{c}\frac{I n\, a\, dz''}{\left[a^2+(z-z'')^2\right]^{3/2}}\, 2\pi a\,\hat{k}$$

$$= \frac{2\pi}{c} I n\,\hat{k}\,\frac{a^2\,dz''}{\left[a^2+(z-z'')^2\right]^{3/2}}$$

(13) $$\vec{B}_{sol}(\rho=0,\phi,z) = \int_{z_1}^{z_2}\vec{B}_{dz''}(\rho=0,\phi,z)$$

$$= \frac{2\pi}{c} I n\,\hat{k}\int_{z_1}^{z_2} dz''\,\frac{a^2}{\left[a^2+(z-z'')^2\right]^{3/2}}$$

$$z-z'' = a\tan\theta$$
$$-dz'' = a\sec^2\theta\, d\theta$$
$$1+\tan^2\theta = \sec^2\theta$$
$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{2\pi}{c} I n\,\hat{k}\int_{\tan^{-1}\frac{z-z_2}{a}}^{\tan^{-1}\frac{z-z_1}{a}}\cos\theta\, d\theta$$

$$= \frac{2\pi}{c} I n\,\hat{k}\left[\frac{z-z_1}{\sqrt{a^2+(z-z_1)^2}} - \frac{z-z_2}{\sqrt{a^2+(z-z_2)^2}}\right]$$

(14) For $z_1 = -L$, $z_2 = +L$ and $z << L$, $a << L$

$$\vec{B}_{sol}(\rho=0,\phi,z) = \frac{4\pi}{c} I n\,\hat{k}$$