Electromagnetism ⑨

## I A Canonical momentum, electric energy & magnetic energy

① $\vec{F} = -\vec{\nabla} W$ work

$\vec{r}_q(t) \rightarrow$ position of the point charge

$\vec{v}_q(t) \rightarrow$ velocity of the point charge

② $\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$

$$\vec{F}(\vec{r}_q,t) = q\,\vec{E}(\vec{r}_q,t) + \frac{q}{c}\vec{v}_q \times \vec{B}(\vec{r}_q,t)$$

$$= \int d^3r'\, q\, \delta^{(3)}(\vec{r}\,'-\vec{r}_q(t))\, \vec{E}(\vec{r}\,',t) + \frac{1}{c}\int d^3r'\, q\, \delta^{(3)}(\vec{r}\,'-\vec{r}_q(t))\, \vec{v}_q(t)\times\vec{B}(\vec{r}\,',t)$$

$$= \int d^3r'\, q\, \delta^{(3)}(\vec{r}\,'-\vec{r}_q(t)) \left[-\vec{\nabla}_{r'}\,\phi(\vec{r}\,',t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{r}\,',t)\right]$$

$$+ \frac{1}{c}\int d^3r'\, q\, \delta^{(3)}(\vec{r}\,'-\vec{r}_q(t))\, \vec{v}_q(t)\times\left[\vec{\nabla}_{r'}\times\vec{A}(\vec{r}\,',t)\right]$$

$$= -\int d^3r'\, q\, \delta^{(3)}(\vec{r}\,'-\vec{r}_q(t))\, \vec{\nabla}_{r'}\,\phi(\vec{r}\,',t)$$

$$-\frac{1}{c}\int d^3r'\, q\, \delta^{(3)}(\vec{r}\,'-\vec{r}_q(t))\, \frac{\partial}{\partial t}\vec{A}(\vec{r}\,',t)$$

$$+\frac{1}{c}\int d^3r'\, q\, \delta^{(3)}(\vec{r}\,'-\vec{r}_q(t))\, \vec{v}_q(t)\times\left[\vec{\nabla}_{r'}\times\vec{A}(\vec{r}\,',t)\right]$$

(3) $$-\int d^3r'\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q)\, \vec{\nabla}_{r'}\, \phi(\vec{r}',t)$$

$$= -\int d^3r'\, \vec{\nabla}_{r'} \left\{ q\,\delta^{(3)}(\vec{r}'-\vec{r}_q)\, \phi(\vec{r}',t) \right\} + \int d^3r' \left[ \vec{\nabla}_{r'}\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q) \right] \phi(\vec{r}',t)$$

$\hookrightarrow = 0$ ($\because$ the surface encloses all the charges)

$$= +\int d^3r' \left[ \vec{\nabla}_{r'}\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q) \right] \phi(\vec{r}',t)$$

$$= -\int d^3r' \left[ \vec{\nabla}_{r_q}\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q) \right] \phi(\vec{r}',t)$$

$$= -\vec{\nabla}_{r_q} \int d^3r'\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q)\, \phi(\vec{r}',t)$$

$$= -\vec{\nabla}_{r_q}\, q\, \phi(\vec{r}_q,t)$$

(4) $$\frac{1}{c}\int d^3r'\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q(t))\, \vec{v}_q(t) \times \left[ \vec{\nabla}_{r'} \times \vec{A}(\vec{r}',t) \right]$$

$$= \frac{1}{c}\, \epsilon^{ijk}\, v_q^j(t) \int d^3r'\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q(t))\, \epsilon^{kmn}\, \nabla_{r'}^m A^n(\vec{r}',t)$$

$$= \frac{1}{c}\left(\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}\right) v_q^j(t) \int d^3r'\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q(t))\, \nabla_{r'}^m A^n(\vec{r}',t)$$

$$= -\frac{1}{c}\left(\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}\right) v_q^j(t) \int d^3r' \left[ \nabla_{r'}^m\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q(t)) \right] A^n(\vec{r}',t)$$

$+$ s. term ($=0$ $\because$ surface encloses all charges)

$$= -\frac{1}{c}\, v_q^j(t) \int d^3r' \left[ \nabla_{r'}^i\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q(t)) \right] A^j(\vec{r}',t)$$

$$+ \frac{1}{c}\, v_q^j(t) \int d^3r' \left[ \nabla_{r'}^j\, q\, \delta^{(3)}(\vec{r}'-\vec{r}_q(t)) \right] A^i(\vec{r}',t)$$

⑤ $$+\frac{1}{c}\, V_q^{\,j}(t) \int d^3r' \left[\nabla_{r'}^{\,j}\, q\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\right] A^i(\vec{r}',t)$$

$$= -\frac{1}{c}\, V_q^{\,j}(t) \int d^3r' \left[\nabla_{r_q}^{\,j}\, q\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\right] A^i(\vec{r}',t)$$

$$= -\frac{1}{c} \int d^3r' \left[\frac{d r_q^{\,j}(t)}{dt} \frac{\partial}{\partial r_q^{\,j}}\, q\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\right] A^i(\vec{r}',t)$$

$$= -\frac{1}{c} \int d^3r' \, \frac{\partial}{\partial t}\left[q\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\right] \vec{A}(\vec{r}',t)$$

⑥ $$-\frac{1}{c}\, V_q^{\,j}(t) \int d^3r' \left[\nabla_{r'}^{\,i}\, q\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\right] A^j(\vec{r}',t)$$

$$= +\frac{1}{c}\, V_q^{\,j}(t)\, \nabla_{r_q}^{\,i} \int d^3r'\, q\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\, A^j(\vec{r}',t)$$

$$\left(\because\ \vec{\nabla}_a \vec{V}_a = \frac{d}{dx_a^i}\frac{dx_a^j}{dt} = \frac{d}{dt}\frac{dx_a^j}{dx_a^i} = \frac{d}{dt}\delta^{ij} = 0\right)$$

$$= +\frac{1}{c}\, \nabla_{r_q}^{\,i}\, V_q^{\,j}(t) \int d^3r'\, q\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\, A^j(\vec{r}',t)$$

$$= \frac{1}{c}\, \nabla_{r_q}^{\,i}\, V_q^{\,j}(t)\, q\, A^j(\vec{r}_q(t),t)$$

$$= -\vec{\nabla}_{r_q}\left[-\frac{q}{c}\, \vec{V}_q \cdot \vec{A}(\vec{r}_q,t)\right]$$

(7) Using (3) to (6) in (2)

$$\vec{F}(\vec{r}_q, t) = -\vec{\nabla}_{r_q}\left[q\,\phi(\vec{r}_q, t)\right] - \vec{\nabla}_{r_q}\left[-\frac{q}{c}\,\vec{v}_q\cdot\vec{A}(\vec{r}_q, t)\right]$$

$$-\frac{1}{c}\int d^3r'\, q\,\delta^{(3)}(\vec{r}'-\vec{r}_q(t))\,\frac{\partial\vec{A}(\vec{r}',t)}{\partial t} - \frac{1}{c}\int d^3r'\,\frac{\partial}{\partial t}\left[q\,\delta^{(3)}(\vec{r}'-\vec{r}_q(t))\right]\vec{A}(\vec{r}',t)$$

$$= -\vec{\nabla}_{r_q}\left[q\,\phi(\vec{r}_q, t) - \frac{q}{c}\,\vec{v}_q\cdot\vec{A}(\vec{r}_q, t)\right]$$

$$-\frac{1}{c}\frac{\partial}{\partial t}\int d^3r'\, q\,\delta^{(3)}(\vec{r}'-\vec{r}_q(t))\,\vec{A}(\vec{r}',t)$$

$$= -\vec{\nabla}_{r_q}\left[q\,\phi(\vec{r}_q, t) - \frac{q}{c}\,\vec{v}_q\cdot\vec{A}(\vec{r}_q, t)\right] - \frac{q}{c}\frac{d}{dt}\vec{A}(\vec{r}_q(t), t)$$

(8) $$\frac{d}{dt} m\vec{v}_q = -\vec{\nabla}_q\left[q\,\phi(\vec{r}_q, t) - \frac{q}{c}\,\vec{v}_q\cdot\vec{A}(\vec{r}_q, t)\right] - \frac{d}{dt}\left(\frac{q}{c}\vec{A}(\vec{r}_q(t), t)\right)$$

$$\frac{d}{dt}\underbrace{\left\{m\vec{v}_q + \frac{q}{c}\vec{A}(\vec{r}_q(t), t)\right\}}_{\text{canonical momentum}} = -\vec{\nabla}_q\Big[\underbrace{q\,\phi(\vec{r}_q, t)}_{\text{electric energy}} - \underbrace{\frac{q}{c}\,\vec{v}_q\cdot\vec{A}(\vec{r}_q, t)}_{\text{magnetic energy}}\Big]$$

Interpretation seems to be wrong
ref: Milton's book, Sec 6.1 } THINK

I B Canonical momentum, electric energy & magnetic energy

(Simpler derivation)

① $\vec{F}(\vec{r}_q,t) = q\,\vec{E}(\vec{r}_q,t) + \frac{q}{c}\vec{v}_q \times \vec{B}(\vec{r}_q,t)$

$$= -q\vec{\nabla}_q\phi(\vec{r}_q,t) - \frac{q}{c}\frac{\partial}{\partial t}\vec{A}(\vec{r}_q,t) + \frac{q}{c}\vec{v}_q \times [\vec{\nabla}_q \times \vec{A}(\vec{r}_q,t)]$$

② $\frac{q}{c}\vec{v}_q \times [\vec{\nabla}_q \times \vec{A}(\vec{r}_q,t)] = \frac{q}{c}\epsilon^{ijk} v_q^j \epsilon^{kmn} \nabla_q^m A^n$

$$= \frac{q}{c}(\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}) v_q^j \nabla_q^m A^n$$

$$= \frac{q}{c} v_q^j \nabla_q^i A^j - \frac{q}{c} v_q^j \nabla_q^j A^i$$

$$= \frac{q}{c}\nabla_q^i (v_q^j A^j) - \frac{q}{c} v_q^j \nabla_q^j A^i$$

Actually $\vec{\nabla}_q$ does not act on $\vec{v}_a$ at all.

$\left(\because \nabla_q^i v_q^j = \frac{d}{dx_q^i}\frac{dx_q^j}{dt} = \frac{d}{dt}\delta^{ij} = 0\right)$

③ $\vec{F}(\vec{r}_q,t) = -q\vec{\nabla}_q\phi(\vec{r}_q,t) + \frac{q}{c}\vec{\nabla}_q(\vec{v}_q\cdot\vec{A}) - \frac{q}{c}\left\{\frac{\partial}{\partial t} + \vec{v}_q\cdot\vec{\nabla}_q\right\}\vec{A}(\vec{r}_q(t),t)$

$$= -\vec{\nabla}_q\left[q\,\phi(\vec{r}_q,t) - \frac{q}{c}\vec{v}_q\cdot\vec{A}(\vec{r}_q,t)\right] - \frac{q}{c}\frac{d}{dt}\vec{A}(\vec{r}_q(t),t)$$

$$\frac{d}{dt}(m\vec{v}_q) = -\vec{\nabla}_q\left[q\,\phi(\vec{r}_q,t) - \frac{q}{c}\vec{v}_q\cdot\vec{A}(\vec{r}_q,t)\right]$$

$$\frac{d}{dt}\underbrace{\left[m\vec{v}_q + \frac{q}{c}\vec{A}(\vec{r}_q(t),t)\right]}_{\text{canonical momentum}} = -\vec{\nabla}_q\Big[\underbrace{q\,\phi(\vec{r}_q,t)}_{\text{electric energy}} - \underbrace{\frac{q}{c}\vec{v}_q\cdot\vec{A}(\vec{r}_q,t)}_{\text{magnetic energy}}\Big]$$

Show that $\vec{\nabla}\times\vec{F} = -\frac{q}{c}\frac{d\vec{B}}{dt}$ & thus electromagnetic is not conservative

Interpretation seems to wrong ref: Sec 6.1, Milton's book } THINK!

## II Magnetostatic energy

① Magnetostatic work done to move a current carrying wire.

$$W_{mag} = -\frac{q}{c}\,\vec{V}_q \cdot \vec{A}(\vec{r}_q)$$

→ Note that there is no time dep. in $\vec{A}$.

$$= -\frac{1}{c}\int d^3r_1\; \vec{J}_1(\vec{r}_1)\cdot\vec{A}(\vec{r}_1)$$

→ Magnetostatics requires steady currents

② Magnetostatic energy required to assemble two current carrying wires:

$$W_{mag} = -\frac{1}{c^2}\int d^3r_1 \int d^3r_2\; \frac{\vec{J}_1(\vec{r}_1)\cdot\vec{J}_2(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|}$$

③ Magnetostatic egy to assemble 'n' current carrying wires:

$$W_{mag} = -\frac{1}{c^2}\,\frac{1}{2}\sum_i\sum_j \int d^3r_i \int d^3r_j\; \frac{\vec{J}_i(\vec{r}_i)\cdot\vec{J}_j(\vec{r}_j)}{|\vec{r}_i-\vec{r}_j|} + \frac{1}{c^2}\sum_i \int d^3r_i \int d^3r_i\; \frac{J_i^2(r_i)}{|\vec{r}_i-\vec{r}_i|}$$

"self-action" work done to create the current wire.

## III Magnetostatic energy of a current loop in an external field:

① $$W_{mag} = -\frac{1}{c}\int d^3r\ \vec{J}(\vec{r})\cdot \underbrace{\vec{A}(\vec{r})}_{\text{due to external field}}$$

$$= -\frac{1}{c}\int d^2r_\perp \int d\xi\ \vec{J}(\vec{r}_\perp,\xi)\cdot\vec{A}(\xi)$$

$$= -\frac{1}{c}\, I \int d\xi\ \hat{n}(\xi)\cdot\vec{A}(\xi)$$

current in magnetostatics are steady & thus $\xi$ independent.

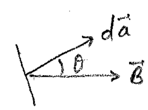
② For a loop

$$W_{mag} = -\frac{1}{c}\, I \oint d\xi\ \hat{n}(\xi)\cdot\vec{A}(\xi)$$

$$= -\frac{1}{c}\, I \int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{A})$$

$$= -\frac{1}{c}\, I \int_S d\vec{a}\cdot\vec{B}$$

$\text{flux} = \int_S d\vec{a}\cdot\vec{B}$




③ If $\vec{B}$ is constant on the surface

$$W_{mag} = -\frac{1}{c}\, I A B\cos\theta$$

④ Work done to move the loop from $\theta_1$ to $\theta_2$.

$$W_{final} - W_{initial} = -\frac{1}{c}\, IAB\left[\cos\theta_2 - \cos\theta_1\right]$$

## IV Inductance

1. We have seen that the magnetostatic energy due to 'n' (steady) current carrying wires is

$$W_{mag} = -\frac{1}{c^2}\frac{1}{2}\sum_i \sum_{\substack{j \\ i\neq j}} \int d^3r_i \int d^3r_j \, \frac{\vec{J}_i(\vec{r}_i)\cdot\vec{J}_j(\vec{r}_j)}{|\vec{r}_i - \vec{r}_j|}$$

$$= -\frac{1}{c^2}\frac{1}{2}\sum_i \sum_{\substack{j \\ i\neq j}} I_i I_j \int d\xi_i \int d\xi_j \, \frac{\hat{n}_i(\xi_i)\cdot\hat{n}_j(\xi_j)}{|\vec{r}_i - \vec{r}_j|}$$

2. Define

$$L_{ij} = \frac{1}{c^2}\int d\xi_i \int d\xi_j \, \frac{\hat{n}_i(\xi_i)\cdot\hat{n}_j(\xi_j)}{|\vec{r}_i - \vec{r}_j|}$$

3. $$W_{mag} = -\frac{1}{2}\sum_i \sum_j L_{ij} I_i I_j$$

4. $\left.\begin{array}{l}\text{Mutual inductance of} \\ \text{two wires}\end{array}\right\} = L_{12} + L_{21} = 2L_{12} \equiv M_{12}$