Electromagnetism ⑩

I Electric dipole moment and magnetic dipole moment :

$\vec{R} \rightarrow$ position of reference pt.
$\vec{V} \rightarrow$ velocity of ref. point.

① Consider 'n' charges

② Charge $= q = \sum_a q_a$

③ Electric Dipole moment $= \vec{d} = \sum_a q_a (\vec{r}_a - \vec{R})$

$$= \sum_a q_a \vec{r}_a \qquad (\text{if } q = 0 \text{ or } \vec{R} = 0)$$

$$= \sum_a \int d^3r' \, \vec{r}' \, q_a \, \delta^{(3)}(\vec{r}' - \vec{r}_a)$$

$$= \int d^3r' \, \vec{r}' \, \rho(\vec{r}')$$

⑤ Magnetic dipole moment $\equiv \vec{\mu} = \frac{1}{2c} \sum_a q_a (\vec{r}_a - \vec{R}) \times (\vec{V}_a - \vec{V})$

$$= \frac{1}{2c} \sum_a q_a \vec{r}_a \times \vec{V}_a \qquad (\text{if } \vec{R} = 0 \text{ \& } \vec{V} = 0)$$

$$= \frac{1}{2c} \sum_a q_a \int d^3r' \, \vec{r}_a \times \vec{V}_a \, \delta^{(3)}(\vec{r}' - \vec{r}_a)$$

$$= \frac{1}{2c} \int d^3r' \, \vec{r}' \times \vec{J}(\vec{r}')$$

II Force on an atom

① Consider a neutral charge distribution

$$\sum_a q_a = 0$$

② $$\vec{F}(t) = \sum_a \left[q_a \vec{E}(\vec{r}_a, t) + \frac{q_a}{c} \vec{v}_a \times \vec{B}(\vec{r}_a, t) \right]$$

$\vec{R}$ is a reference point (say center of mass).

③ Approximate :

$$\vec{E}(\vec{r}_a, t) = \vec{E}(\vec{R}, t) + (\vec{r}_a - \vec{R}) \cdot \vec{\nabla}_R \vec{E}(\vec{R}, t) + \dots$$

$$\vec{B}(\vec{r}_a, t) = \vec{B}(\vec{R}, t) + (\vec{r}_a - \vec{R}) \cdot \vec{\nabla}_R \vec{B}(\vec{R}, t) + \dots$$

④ $$\vec{F}(t) = \left(\sum_a q_a\right) \vec{E}(\vec{R}, t) + \sum_a q_a (\vec{r}_a - \vec{R}) \cdot \vec{\nabla}_R \vec{E}(\vec{R}, t)$$
$$+ \left(\sum_a \frac{q_a}{c} \vec{v}_a\right) \times \vec{B}(\vec{R}, t) + \sum_a \frac{q_a}{c} \vec{v}_a \times \left\{(\vec{r}_a - \vec{R}) \cdot \vec{\nabla}_R\right\} \vec{B}(\vec{R}, t) + \dots$$

$\downarrow$ ref: page 34, J. Schwinger's book.

$$= - \nabla_R \left[\vec{d} \cdot \vec{E}(\vec{R}, t) + \vec{\mu} \cdot \vec{B}(\vec{R}, t) \right] + \frac{1}{c} \frac{d}{dt} \left(\vec{d} \times \vec{B}(\vec{R}, t) \right)$$

Note : $\vec{d}$ & $\vec{\mu}$ are $\vec{R}$ independent.

III Force on a macroscopic body

① Using II, for a macroscopic body whose atomic density is $n(\vec{r})$, the total force is

$$\vec{F}(t) = \int d^3r \; n(\vec{r}) \left[\vec{F}_{atom}(\vec{r}, t) \right]$$

where

$$\vec{F}_{atom}(\vec{r},t) = d(\vec{r}) \times \left(\vec{\nabla} \times \vec{E}(\vec{r})\right) + \left(\vec{d}(\vec{r}) \cdot \vec{\nabla}\right) \vec{E}(\vec{r})$$
$$+ \vec{\mu}(\vec{r}) \times \left(\vec{\nabla} \times \vec{B}(\vec{r})\right) + \left(\vec{\mu}(\vec{r}) \cdot \vec{\nabla}\right) \vec{B}(\vec{r})$$
$$+ \frac{d}{dt}\left(\frac{1}{c} \, d(\vec{r}) \times \vec{B}(\vec{r})\right)$$

② Note that in the earlier form $n(\vec{r})$ could not be combined with $d(\vec{r})$ & $\mu(\vec{r})$ because of the gradient.

③
$$\vec{P}(\vec{r},t) = n(\vec{r}) \; d(\vec{r},t)$$
$$\vec{M}(\vec{r},t) = n(\vec{r}) \; \vec{\mu}(\vec{r},t)$$

$\downarrow$ ref: page 38, J. Schwinger's book.

④
$$\vec{F}(t) = \int d^3r \left[-(\vec{\nabla} \cdot \vec{P}) \, \vec{E} + \frac{1}{c} \left\{ \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times M \right\} \times \vec{B} \right]$$

$$\rho_{eff}(\vec{r},t) = - \vec{\nabla} \cdot \vec{P}(\vec{r},t)$$
$$\vec{J}_{eff}(\vec{r},t) = \frac{\partial}{\partial t} \vec{P}(\vec{r},t) + \vec{\nabla} \times \vec{M}(\vec{r},t)$$

(5) Thus we can write the macroscopic Maxwell's equ to be

$$\vec{\nabla}\cdot\vec{E} = 4\pi\,(\rho_{free} + \rho_{eff})$$

$$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\left(\vec{J}_{free} + \vec{J}_{eff}\right)$$

(6)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_{free} - 4\pi\,\vec{\nabla}\cdot\vec{P}$$

$$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}_{free} + \frac{4\pi}{c}\left\{\frac{\partial\vec{P}}{\partial t} + \vec{\nabla}\times M\right\}$$

(7)

$$\vec{D} \equiv \vec{E} + 4\pi\vec{P}$$

$$\vec{H} \equiv \vec{B} - 4\pi\vec{M}$$

(8)

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

$$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

IV Time-independent point electric dipole moment ($-\vec\nabla\cdot\vec P$ term)

① For a point electric dipole moment we have.

$$\vec P(\vec r, t) = \vec P_0(t)\, \delta^{(3)}(\vec r - \vec r_p)$$

$$= \vec P_0\, \delta^{(3)}(\vec r - \vec r_p)$$

↳ time-independence

$$\ominus \xrightarrow{\vec p} \oplus$$

$$\rho = +q\,\delta(x-x_0)\,\delta(y)\,\delta(z) - q\,\delta(x)\,\delta(y)\,\delta(z)$$

$$= q(-x_0)\left[\frac{\delta(x-x_0)-\delta(x)}{(-x_0)}\right]\delta(y)\delta(z)$$

$$= -q\,x_0\,\frac{d}{dx}\,\delta^{(3)}(\vec r)$$

$$= -\vec p\cdot\vec\nabla\,\delta^{(3)}(\vec r)$$

② Charge density

$$\rho_{eff}(\vec r, t) = -\vec\nabla\cdot\vec P(\vec r, t)$$

$$= -\vec P_0\cdot\vec\nabla_r\,\delta^{(3)}(\vec r - \vec r_p)$$

③ Electric potential & electric field

$$-\nabla^2\phi(\vec r, t) = -\vec\nabla\cdot\vec P$$

$$= -\vec P_0\cdot\vec\nabla_r\,\delta^{(3)}(\vec r - \vec r_p)$$

$$\phi(\vec r, t) = \int d^3r'\,\frac{\left[(-1)\,\vec P_0\cdot\vec\nabla_{r'}\,\delta^{(3)}(\vec r' - \vec r_p)\right]}{|\vec r - \vec r'|}$$

$$= -\vec P_0\cdot\int d^3r'\,\frac{1}{|\vec r - \vec r'|}\,\vec\nabla_{r'}\,\delta^{(3)}(\vec r' - \vec r_p)$$

$$= +\vec P_0\cdot\vec\nabla_{r_p}\int d^3r'\,\frac{\delta^{(3)}(\vec r' - \vec r_p)}{|\vec r - \vec r'|}$$

$$= \vec P_0\cdot\vec\nabla_{r_p}\,\frac{1}{|\vec r - \vec r_p|}$$

(4)

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi(\vec{r},t)$$

$$= -\vec{\nabla}_r \left[\vec{P}_0 \cdot \vec{\nabla}_{r_p} \frac{1}{|\vec{r}-\vec{r}_p|} \right]$$

$$= \left(\vec{P}_0 \cdot \vec{\nabla}_{r_p}\right) \vec{\nabla}_{r_p} \frac{1}{|\vec{r}-\vec{r}_p|}$$

(5) Force on a point electric dipole moment in an external electric field

$$\vec{F} = \int d^3r \; \rho(\vec{r}) \; \vec{E}_{ext}(\vec{r})$$

$$= \int d^3r \left[-\vec{P}_0 \cdot \vec{\nabla}_r \; \delta^{(3)}(\vec{r}-\vec{r}_p) \right] \vec{E}_{ext}(\vec{r})$$

$$= -\vec{P}_0 \cdot \int d^3r \left[\vec{\nabla}_r \; \delta^{(3)}(\vec{r}-\vec{r}_p) \right] \vec{E}_{ext}(\vec{r})$$

$$= +\vec{P}_0 \cdot \vec{\nabla}_{r_p} \vec{E}_{ext}(\vec{r}_p)$$

(6) Torque on a point electric dipole moment in an external electric field

$$\vec{\tau} = \int d^3r \; \rho(\vec{r}) \; \vec{r} \times \vec{E}_{ext}(\vec{r})$$

$$= \int d^3r \left[-\vec{P}_0 \cdot \vec{\nabla}_r \; \delta^{(3)}(\vec{r}-\vec{r}_p) \right] \vec{r} \times \vec{E}_{ext}(\vec{r})$$

$$= +\left(\vec{P}_0 \cdot \vec{\nabla}_{r_p}\right) \vec{r}_p \times \vec{E}_{ext}(\vec{r}_p)$$

$$= \vec{P}_0 \times \vec{E}_{ext}(\vec{r}_p) + \vec{r}_p \times \left[(\vec{P}_0 \cdot \vec{\nabla}_{r_p}) \vec{E}_{ext}(\vec{r}_p) \right]$$

V. Time-dependent point electric dipole moment ($\frac{\partial \vec{P}}{\partial t}$ term)

① $\vec{P}(\vec{r},t) = \vec{P}_0(t)\, \delta^{(3)}(\vec{r} - \vec{r}_p(t))$

$\vec{P}_0(t)$: intrinsic time dependence

$\delta^{(3)}(\vec{r} - \vec{r}_p(t))$: time dependence due to motion.

② $$\vec{J}_{eff}(\vec{r},t) = \frac{\partial \vec{P}(\vec{r},t)}{\partial t}$$
$$= \left[\frac{\partial}{\partial t}\vec{P}_0(t)\right] \delta^{(3)}(\vec{r} - \vec{r}_p(t)) + \vec{P}_0(t)\, (\vec{V}_p \cdot \vec{\nabla}_{r_p})\, \delta^{(3)}(\vec{r} - \vec{r}_p(t))$$

③ If $\vec{J}_{eff}(\vec{r},t)$ has no time dependence, then we have magnetostatics.

$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} J_{eff}(\vec{r})$$

④ Eg: (i) $\vec{r}_p(t)$ is constant in time.

(ii) $\vec{P}_0(t) = \vec{\alpha}_0 t + \vec{\beta}_0$

← This is not magnetostatics because $\frac{\partial \vec{E}}{\partial t} \neq 0$

VI Point magnetic dipole moment ($\vec{\nabla} \times \vec{M}$ term)

① $$\vec{M}(\vec{r},t) = \vec{M}_0(t)\, \delta^{(3)}(\vec{r} - \vec{r}_m(t))$$
$$= \vec{M}_0\, \delta^{(3)}(\vec{r} - \vec{r}_m) \qquad \text{(if time independent)}$$

② $$\vec{J}_{eff}(\vec{r},t) = \vec{\nabla} \times [\vec{M}_0(t)\, \delta^{(3)}(\vec{r} - \vec{r}_m(t))]$$
$$= - \vec{M}_0(t) \times \vec{\nabla}_r\, \delta^{(3)}(\vec{r} - \vec{r}_m(t))$$

③ For time-independent case

$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} [- \vec{M}_0 \times \vec{\nabla}_r\, \delta^{(3)}(\vec{r} - \vec{r}_m)]$$

④ $$\vec{A}(\vec{r},t) = \frac{1}{c} \int d^3r' \frac{[- \vec{M}_0 \times \vec{\nabla}_{r'}\, \delta^{(3)}(\vec{r}' - \vec{r}_m)]}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{c}\, \vec{M}_0 \times \vec{\nabla}_{r_m} \frac{1}{|r - \vec{r}_m|}$$

⑤ $$\vec{B}(\vec{r},t) = \vec{\nabla}_r \times \vec{A}(\vec{r},t)$$
$$= - \frac{1}{c}\, \vec{\nabla}_{r_m} \times (\vec{M}_0 \times \vec{\nabla}_{r_m}) \frac{1}{|\vec{r} - \vec{r}_m|}$$

(6) Force on a point magnetic dipole moment in an external magnetic field

$$\vec{F} = \int d^3r \; \vec{J}(\vec{r}) \times \vec{B}_{ext}(\vec{r})$$

$$= \int d^3r \left[-\vec{M}_0 \times \vec{\nabla}_r \, \delta^{(3)}(\vec{r}-\vec{r}_m) \right] \times \vec{B}_{ext}(\vec{r})$$

$$= \left(\vec{M}_0 \times \vec{\nabla}_{r_m} \right) \times \underline{\underline{\vec{B}_{ext}(\vec{r}_m)}}$$

(7) Torque on a point magnetic dipole moment in an external magnetic field

$$\vec{\tau} = \int d^3r \; \vec{r} \times \left[\vec{J}(\vec{r}) \times \vec{B}_{ext}(\vec{r}) \right]$$

$$= \int d^3r \; \vec{r} \times \left[\left\{ -\vec{M}_0 \times \vec{\nabla}_r \, \delta^{(3)}(\vec{r}-\vec{r}_m) \right\} \times \vec{B}_{ext}(\vec{r}) \right]$$

$$= \underline{\underline{\vec{r}_m \times \left[\left(\vec{M}_0 \times \vec{\nabla}_{r_m} \right) \times \vec{B}_{ext}(\vec{r}_m) \right]}}$$