Electromagnetism 11

I Polarized object of finite size :

$$\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

1. $$\vec{P}(\vec{r},t) = \vec{P}(\vec{r})\ \theta(a-\xi)$$

where $\xi(x,y,z) = a$ is the eqn. defining the boundary of the polarized object.

2. $$\rho_{eff}(\vec{r}) = -\vec{\nabla}\cdot\vec{P}$$
$$= -\left[\vec{\nabla}\cdot\vec{P}(\vec{r})\right]\theta(a-\xi) - \vec{P}(\vec{r})\cdot\vec{\nabla}\,\theta(a-\xi)$$
$$= -\left[\vec{\nabla}\cdot\vec{P}(\vec{r})\right]\theta(a-\xi) + \vec{P}(\vec{r})\cdot\left[\vec{\nabla}\xi(\vec{r})\right]\delta(a-\xi)$$
$$= -\left[\vec{\nabla}\cdot\vec{P}(\vec{r})\right]\theta(a-\xi) + \vec{P}(\vec{r})\cdot\hat{n}_\xi\ \delta(a-\xi)$$
$$= -\left[\vec{\nabla}\cdot\vec{P}(\vec{r})\right]\theta(a-\xi) + \cdots$$

($\hat{n} = \vec{\nabla}\xi$ ref: H.W) Try to prove this!

→ volume charge
→ contributes to $\vec{E}$ both inside & outside the surface.

→ surface charge ('bound charge')
→ does not contribute to $\vec{E}$ inside the surface.

II Magnetized object of finite size:

1. $$\vec{M}(\vec{r},t) = \vec{M}(\vec{r})\ \theta(a-\xi)$$

2. $$\begin{aligned}\vec{J}_{eff}(\vec{r}) &= \vec{\nabla}\times\vec{M} \\ &= \epsilon^{ijk}\ \nabla^j\left[M^k(\vec{r})\ \theta(a-\xi)\right] \\ &= \epsilon^{ijk}\left[\nabla^j M^k(\vec{r})\right]\theta(a-\xi) + \epsilon^{ijk}\ M^k(\vec{r})\ \nabla^j\theta(a-\xi) \\ &= \left[\vec{\nabla}\times\vec{M}(\vec{r})\right]\theta(a-\xi) - \left[\vec{\nabla}\xi(\vec{r})\right]\times\vec{M}(\vec{r})\ \delta(a-\xi) \\ &= \underbrace{\left[\vec{\nabla}\times\vec{M}(\vec{r})\right]\theta(a-\xi)}_{\text{volume current}} - \underbrace{\left[\hat{n}\times\vec{M}(\vec{r})\right]\delta(a-\xi)}_{\text{surface current}}\end{aligned}$$

III Uniformly polarized sphere :

$$r = \sqrt{x^2+y^2+z^2}$$

① $\vec{P}(\vec{r},t) = \vec{P}_0 \; \theta(a-r)$

②
$$\begin{aligned}\rho_{eff}(\vec{r},t) &= -\vec{\nabla}\cdot\vec{P} \\ &= -\vec{P}_0 \cdot \vec{\nabla}\,\theta(a-r) \\ &= -\vec{P}_0 \cdot \delta(a-r)\,(-1)\,\frac{\vec{r}}{r} \\ &= +\left[\vec{P}_0\cdot\hat{r}\right]\delta(a-r)\end{aligned}$$

③
$$-\nabla^2\phi(\vec{r}) = 4\pi\,\rho(\vec{r})$$

$$\begin{aligned}\phi(\vec{r}) &= \int d^3r' \,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \\ &= \int d^3r' \,\frac{[\vec{P}_0\cdot\hat{r}']\,\delta(a-r')}{[r^2+r'^2-2rr'\cos\theta']^{1/2}} \\ &= \int_0^\infty r'^2dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \,\frac{\delta(a-r')\,\vec{P}_0\cdot[\sin\theta'\cos\phi'\,\hat{i} + \sin\theta'\sin\phi'\,\hat{j} + \cos\theta'\,\hat{k}]}{[r^2+r'^2-2rr'\cos\theta']^{1/2}} \\ &= 2\pi a^2\,(\vec{P}_0\cdot\hat{k}) \int_0^\pi \sin\theta' d\theta' \,\frac{\cos\theta'}{[r^2+a^2-2ra\cos\theta']^{1/2}}\end{aligned}$$

(choosing $\vec{r}$ along $\hat{z}$) uses symmetry argument.

where we have used $\int_0^{2\pi} d\phi' \cos\phi' = 0$ & $\int_0^{2\pi} d\phi' \sin\phi' = 0$.

④ Using problem ②(d) in H·w # ⑦

$$\int_0^\pi \sin\theta' d\theta' \frac{\cos\theta'}{\left[r^2 + a^2 - 2ar\cos\theta'\right]^{1/2}} = \begin{cases} \frac{2}{3}\frac{r}{a^2} & r < a \\ \frac{2}{3}\frac{a}{r^2} & r > a \end{cases}$$

⑤ Using ④ in ③

$$\phi(\vec{r}) = \begin{cases} \frac{4\pi}{3}(\vec{P}_0\cdot\hat{k})\, r & r < a \\ \frac{4\pi}{3}a^3(\vec{P}_0\cdot\hat{k})\frac{1}{r^2} & r > a \end{cases}$$

$$= \begin{cases} \frac{4\pi}{3}(\vec{P}_0\cdot\hat{r})\, r & r < a \\ \frac{4\pi}{3}a^3\frac{(\vec{P}_0\cdot\hat{r})}{r^2} & r > a \end{cases}$$

(where we have released the choice we made of $\hat{r}$ to be along $\hat{k}$)

⑥ $$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

$$= \begin{cases} -\frac{4\pi}{3}\left[\vec{\nabla}(\vec{P}_0\cdot\vec{r})\right] & r < a \\ -\frac{4\pi}{3}a^3\left[\vec{\nabla}(\vec{P}_0\cdot\vec{r})\right]\frac{1}{r^3} - \frac{4\pi}{3}a^3(\vec{P}_0\cdot\vec{r})\left[\vec{\nabla}\frac{1}{r^3}\right] & r > a \end{cases}$$

$$= \begin{cases} -\frac{1}{3}4\pi\vec{P}_0 & r < a \\ \frac{4\pi}{3}a^3\frac{1}{r^3}\left[3(\vec{P}_0\cdot\hat{r})\hat{r} - \vec{P}_0\right] & r > a \end{cases}$$

(ref: prob ② in H·w # ⑪)

IV Uniformly magnetized sphere :

① $$\vec{M}(\vec{r},t) = \vec{M}_0\ \theta(a-r)$$

② $$\vec{J}_{eff}(\vec{r}) = \vec{\nabla}\times\vec{M}$$
$$= \epsilon^{ijk}\ \nabla^j\left[M_0^k\ \theta(a-r)\right]$$
$$= \epsilon^{ijk}\ M_0^k\ \delta(a-r)\ (-1)\ \hat{r}^j$$
$$= \left[\vec{M}_0\times\hat{r}\right]\ \delta(a-r)$$

③ $$-\nabla^2\vec{A}(\vec{r}) = \frac{4\pi}{c}\ \vec{J}(\vec{r})$$

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r'\ \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

(choosing $\hat{r}$ along $\hat{z}$)

$$= \frac{1}{c}\int d^3r'\ \frac{[\vec{M}_0\times\hat{r}']\ \delta(a-r')}{[r^2+r'^2-2rr'\cos\theta']^{1/2}}$$

$$= \frac{1}{c}\int_0^\infty r'^2dr'\int_0^\pi \sin\theta'd\theta'\int_0^{2\pi}d\phi'\ \frac{\delta(a-r)\ \vec{M}_0\times[\sin\theta'\cos\phi'\,\hat{i}+\sin\theta'\sin\phi'\,\hat{j}+\cos\theta'\,\hat{k}]}{[r^2+r'^2-2rr'\cos\theta']^{1/2}}$$

(using $\int_0^{2\pi}d\phi'\cos\phi'=0$, $\int_0^{2\pi}d\phi'\sin\phi'=0$)

$$= \frac{2\pi a^2}{c}\ \vec{M}_0\times\hat{k}\int_0^\pi \sin\theta'd\theta'\ \frac{\cos\theta'}{[r^2+a^2-2ar\cos\theta']^{1/2}}$$

(using III - ④)

$$= \frac{2\pi a^2}{c}(\vec{M}_0\times\hat{k})\begin{cases}\frac{2}{3}\frac{r}{a^2} & r<a\\ \frac{2}{3}\frac{a}{r^2} & r>a.\end{cases}$$

(releasing the choice of $\hat{r}$ to be along $\hat{k}$)

$$= \begin{cases}\frac{1}{c}\frac{4\pi}{3}(\vec{M}_0\times\hat{r})\,r\\ \frac{1}{c}\frac{4\pi}{3}a^3\frac{(\vec{M}_0\times\hat{r})}{r^2}\end{cases}$$

4 $$\vec{A}(\vec{r}) = \begin{cases} \frac{1}{c} \frac{4\pi}{3} (\vec{M}_0 \times \vec{r}) & r < a \\ \frac{1}{c} \frac{4\pi}{3} a^3 \frac{(\vec{M}_0 \times \vec{r})}{r^3} & r > a \end{cases}$$

5 $$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$$

$$= \begin{cases} \frac{1}{c} \frac{2}{3} 4\pi \vec{M}_0 & r < a \\ \frac{1}{c} \frac{4\pi}{3} a^3 \frac{2\vec{M}_0}{r^3} & r > a \end{cases}$$

(ref: prob 3 H.W #11)