I Potentials

① $\vec{\nabla} \cdot \vec{E} = 4\pi \rho$ $\qquad$ $\vec{\nabla} \cdot \vec{B} = 0$

$-\vec{\nabla} \times \vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\vec{B}$ $\qquad$ $\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$

② $\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

③ $\vec{\nabla} \cdot \left[-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right] = 4\pi \rho$

$-\nabla^2 \phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A} = 4\pi \rho$

④ $\vec{\nabla} \times (\vec{\nabla} \times A) = \frac{1}{c}\frac{\partial}{\partial t}\left[-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \right] + \frac{4\pi}{c}\vec{J}$

$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \frac{1}{c}\vec{\nabla}\frac{\partial \phi}{\partial t}$

⑤ $-\nabla^2 \phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = 4\pi \rho$

$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \frac{4\pi}{c}\vec{J} - \vec{\nabla}\left[\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} \right]$

II Gauges

① $\vec{B}$ and $\vec{E}$ in I-② and thus the Maxwell's equations are invariant under

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda(\vec{x},t)$$

$$\phi \to \phi - \frac{\partial}{\partial t}\lambda(\vec{x},t)$$

② Radiation gauge:

$$\vec{\nabla}\cdot\vec{A} = 0$$

$$-\nabla^2\phi = 4\pi\rho$$

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \frac{4\pi}{c}\vec{J} - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\phi)$$

(i) $\longrightarrow$ $\phi(\vec{x},t) = \int d^3x' \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$

(ii) $\longrightarrow$ $\vec{J}_{eff}(\vec{x},t) = \vec{J} - \frac{1}{4\pi}\frac{\partial}{\partial t}\vec{\nabla}\phi$

(iii) $\longrightarrow$ thus the problem reduces to solving

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \frac{4\pi}{c}\vec{J}_{eff}$$

③ Lorentz gauge:

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$$

$$-\left(\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 4\pi\rho$$

$$-\left(\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \frac{4\pi}{c}\vec{J}$$

III Green's function in the Lorentz gauge

① $$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 4\pi\rho$$

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \frac{4\pi}{c}\vec{J}$$

② Define

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x}-\vec{x}', t-t') = 4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')\,\delta(t-t')$$

③ $$\phi(\vec{x},t) = \int dt' \int d^3x'\; G(\vec{x}-\vec{x}', t-t')\;\rho(\vec{x}',t)$$

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \int dt'\, G(\vec{x}-\vec{x}', t-t')\;\vec{J}(\vec{x}',t)$$

④ $$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

IV Solution to Green's function in the Lorentz gauge

① $$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}-\vec{r}\,', t-t') = 4\pi\, \delta^{(3)}(\vec{r}-\vec{r}\,', t-t')$$

② $$G(\vec{r}-\vec{r}\,', t-t') = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{i\omega(t-t')}\, \tilde{G}(\vec{r}-\vec{r}\,', \omega)$$

$$\delta(x) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{i\omega x}$$

③ Using ② in ①

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{G}(\vec{r}-\vec{r}\,', \omega) = 4\pi\, \delta^{(3)}(\vec{r}-\vec{r}\,')$$

④ $\tilde{G}$ is spherically symmetric, thus choose $\vec{r}\,' = 0$

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{G}(\vec{r}, \omega) = 4\pi\, \delta^{(3)}(\vec{r})$$

⑤ Integrating over a sphere of radius r_s.

$$-\int d^3r\, \vec{\nabla}\cdot\vec{\nabla}\tilde{G} - \frac{\omega^2}{c^2}\int d^3r\, \tilde{G} = 4\pi$$

$$-4\pi r_s^2 \left\{\frac{d}{dr}\tilde{G}(r,\omega)\right\}_{r=r_s} - \frac{\omega^2}{c^2}\int d^3r\, \tilde{G}(r,\omega) = 4\pi$$

⑥ For the static case $(\omega = 0)$

$$\tilde{G}(r, \omega=0) = \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}$$

then argue that

$$\tilde{G}(r,\omega) \sim \frac{1}{r}$$

(7) Using (6) conclude that

$$\int_{r_s} d^3r \; \tilde{G}(r,\omega) \sim 4\pi \int_0^{r_s} r^2 dr \frac{1}{r}$$

$$\sim 2\pi r_s^2$$

$$= 0 \quad \text{in the limit } r_s \to 0$$

(8) Taking the limit $r_s \to 0$ in (5) conclude

$$-4\pi r_s^2 \left\{ \frac{d}{dr} \tilde{G}(r,\omega) \right\}_{r = r_s \to 0} = 4\pi$$

(9) Using $\nabla^2 = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr}$ for the spherical case

our problem reduces to solving

$$\left[\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + \frac{\omega^2}{c^2} \right] \tilde{G}(r,\omega) = 0 \qquad r > 0$$

under the boundary condition

$$\mathop{Lt}_{r \to 0} \; r^2 \frac{d}{dr} \tilde{G}(r,\omega) = -1$$

(10) Substitute

$$\tilde{G}(r,\omega) = \frac{1}{r} \tilde{g}(r,\omega)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \frac{\tilde{g}}{r} \right) + \frac{\omega^2}{c^2} \frac{\tilde{g}}{r} = 0 \qquad \text{for } r > 0$$

$$\frac{d^2}{dr^2} \tilde{g}(r,\omega) = -\frac{\omega^2}{c^2} \tilde{g}(r,\omega) \qquad \text{for } r > 0$$

11 Thus we have

$$\tilde{g}(r,\omega) = A\, e^{+i\frac{\omega}{c}r} + B\, e^{-i\frac{\omega}{c}r} \qquad \text{for} \quad r>0$$

12 Using 11 in 10

$$\tilde{G}(r,\omega) = \frac{A}{r}\, e^{+i\frac{\omega}{c}r} + \frac{B}{r}\, e^{-i\frac{\omega}{c}r} \qquad r>0$$

13 Using 12 in 9

$$\operatorname*{Lt}_{r\to 0}\; r^2\left[-\frac{A}{r^2}\, e^{+i\frac{\omega}{c}r} + i\frac{\omega}{c}\frac{A}{r}\, e^{+i\frac{\omega}{c}r} - \frac{B}{r^2}\, e^{-i\frac{\omega}{c}r} - i\frac{\omega}{c}\frac{B}{r}\, e^{-i\frac{\omega}{c}r}\right] = -1$$

$$-(A+B) = -1$$

$$A+B = 1$$

14 Thus we have.

$$\tilde{G}(\vec{r}-\vec{r}\,',\omega) = \frac{A}{|\vec{r}-\vec{r}\,'|}\, e^{+i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|} + \frac{B}{|\vec{r}-\vec{r}\,'|}\, e^{-i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|} \qquad A+B=1$$

15 Using 14 in 2

$$\tilde{G}(\vec{r}-\vec{r}\,',t-t') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, \tilde{G}(\vec{r}-\vec{r}\,',\omega)$$

$$= \frac{A}{|\vec{r}-\vec{r}\,'|}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega(t-t') + i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}$$

$$+ \frac{B}{|\vec{r}-\vec{r}\,'|}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega(t-t') - i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}$$

(16) Using $\delta(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx}$ we have.

$$\tilde{G}(\vec{r}-\vec{r}', t-t') = \frac{A}{|\vec{r}-\vec{r}'|} \delta\left[(t-t') - \frac{1}{c}|\vec{r}-\vec{r}'|\right] + \frac{B}{|\vec{r}-\vec{r}'|} \delta\left[(t-t') + \frac{1}{c}|\vec{r}-\vec{r}'|\right]$$

with $A+B=1$.

(17) Interpretation :

let $\vec{r}' = 0$ (position of a point charge)

At $t' = t'$ say the charge dissappears!

$\vec{r} \to$ observation point in space

$t \to$ observation point in time

Question : At what time 't' will the observer at point $\vec{r}$ know about the dissappearance?

A term $\to$ $t = t' + \frac{r}{c}$

B term $\to$ $t = t' - \frac{r}{c}$

(18) Thus causality requires $B=0$. Thus $A=1$.

$$G(\vec{r}-\vec{r}', t-t') = \frac{1}{|\vec{r}-\vec{r}'|} \delta\left[(t-t') - \frac{1}{c}|\vec{r}-\vec{r}'|\right].$$

(electromagnetic signals travel at the speed of light.)

V Potentials in terms of the Green's function:

① $$G(\vec{r}-\vec{r}', t-t') = \frac{1}{|\vec{r}-\vec{r}'|}\,\delta\left[(t-t') - \frac{1}{c}|\vec{r}-\vec{r}'|\right]$$

② $$\phi(\vec{r},t) = \int d^3r' \int dt' \frac{1}{|\vec{r}-\vec{r}'|}\,\delta\left[(t-t') - \frac{1}{c}|\vec{r}-\vec{r}'|\right] \rho(\vec{r}',t')$$

$$= \int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\,\rho\left(\vec{r}', t - \frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r' \int dt' \frac{1}{|\vec{r}-\vec{r}'|}\,\delta\left[(t-t') - \frac{1}{c}|\vec{r}-\vec{r}'|\right] \vec{J}(\vec{r}',t')$$

$$= \frac{1}{c}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|}\,\vec{J}\left(\vec{r}', t - \frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

③ $$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$