Electromagnetism (14)

I Radiation

① We have seen

$$\phi(\vec{r},t) = \int d^3r' \int dt' \frac{1}{|\vec{r}-\vec{r}'|} \delta\left[t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right] \rho(\vec{r}',t')$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r' \int dt' \frac{1}{|\vec{r}-\vec{r}'|} \delta\left[t-t'-\frac{1}{c}|\vec{r}-\vec{r}'|\right] \vec{J}(\vec{r}',t')$$

② Note that $\vec{r}'$ is a dummy integration variable and thus has no t or t' dependence. Only for a point particle, after we integrate over $\vec{r}'$ does $\vec{r}_q(t)$ take over the role of $\vec{r}'$. With this observation it is possible to integrate over t' and get

$$\phi(\vec{r},t) = \int d^3r' \frac{1}{|\vec{r}-\vec{r}'|} \rho\left(\vec{r}', t-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r' \frac{1}{|\vec{r}-\vec{r}'|} \vec{J}\left(\vec{r}', t-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

③ <u>Case (i)</u> $|\vec{r} - \vec{r}'| << cT \longrightarrow$ a characteristic time involved in the process

In this case

$$t - \frac{|\vec{r} - \vec{r}'|}{c} \simeq t$$

and thus we get

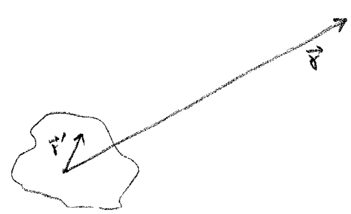
$$\phi(\vec{r}, t) = \int d^3r' \, \frac{1}{|\vec{r} - \vec{r}'|} \, \rho(\vec{r}', t)$$

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int d^3r' \, \frac{1}{|\vec{r} - \vec{r}'|} \, \vec{J}(\vec{r}', t)$$

which is very close to statics.

<u>Case (ii)</u> $|\vec{r}| >> |\vec{r}'|$ (RADIATION)

Observation point $\vec{r}$ is very far from source distribution $\vec{r}'$.



$$|\vec{r} - \vec{r}'| = \sqrt{r^2 + r'^2 - 2\, \vec{r} \cdot \vec{r}'}$$

$$= r \sqrt{1 + \frac{r'^2}{r^2} - 2 \frac{\hat{r} \cdot \vec{r}'}{r}}$$

$$= r \left[1 - \frac{\hat{r} \cdot \vec{r}'}{r} + O\left(\frac{1}{r^2}\right) \right]$$

④ Thus in the regime of radiation

$$|\vec{r} - \vec{r}'| = r - \hat{r}\cdot\vec{r}' + O\left(\tfrac{1}{r}\right)$$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{\left[r - \hat{r}\cdot\vec{r}' + O\left(\frac{1}{r}\right)\right]}$$

$$= \frac{1}{r}\,\frac{1}{\left[1 - \frac{\hat{r}\cdot\vec{r}'}{r} + O\left(\frac{1}{r^2}\right)\right]}$$

$$= \frac{1}{r}\left[1 + \frac{\hat{r}\cdot\vec{r}'}{r} + O\left(\frac{1}{r^2}\right)\right]$$

$$= \frac{1}{r} + O\left(\frac{1}{r^2}\right)$$

$$t_r \equiv t - \frac{1}{c}|\vec{r} - \vec{r}'| = t - \frac{1}{c}\left[r - \hat{r}\cdot\vec{r}' + O\left(\tfrac{1}{r}\right)\right]$$

$$= t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c} + O\left(\tfrac{1}{r}\right)$$

⑤ $$\phi(\vec{r},t) = \int d^3r' \,\frac{1}{|\vec{r}-\vec{r}'|}\,\rho\left(\vec{r}', t - \tfrac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$$= \frac{1}{r}\int d^3r' \,\rho\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right) + O\left(\frac{1}{r^2}\right)$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\,\frac{1}{r}\int d^3r' \,\vec{J}\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right) + O\left(\frac{1}{r^2}\right)$$

II Radiation fields

① $$\phi(\vec{r},t) = \frac{1}{r}\int d^3r'\, \rho\left(\vec{r}',\, t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right) + O\left(\frac{1}{r^2}\right)$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\,\frac{1}{r}\int d^3r'\, \vec{J}\left(\vec{r}',\, t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right) + O\left(\frac{1}{r^2}\right)$$

② $$\vec{E}(\vec{r},t) = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A} \qquad\qquad t_r \equiv t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}$$

③ $$\vec{\nabla}\left[\frac{1}{r}\, f\left(t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right)\right]$$

$$= \left[\vec{\nabla}\frac{1}{r}\right] f\left(t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right) + \frac{1}{r}\left[\vec{\nabla} t_r\right]\left\{\frac{\partial}{\partial t} f(t)\right\}_{t=t_r}$$

$$= -\frac{\hat{r}}{r^2}\, f(t_r) + \frac{1}{r}\left\{\frac{\partial f(t)}{\partial t}\right\}_{t=t_r}\left[-\frac{1}{c}\vec{\nabla} r + \frac{1}{c}\vec{\nabla}(\hat{r}\cdot\vec{r}')\right]$$

④ $$\vec{\nabla}(\hat{r}\cdot\vec{r}') = \frac{1}{r}\left[\vec{r}' - \hat{r}(\hat{r}\cdot\vec{r}')\right]$$

$$= -\frac{1}{r}\,\hat{r}\times(\hat{r}\times\vec{r}')$$

⑤ Using ④ in ③

$$\vec{\nabla}\left[\frac{1}{r}\, f\left(t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}\right)\right] = -\frac{1}{c}\,\frac{\hat{r}}{r}\left\{\frac{\partial f(t)}{\partial t}\right\}_{t=t_r} + O\left(\frac{1}{r^2}\right)$$

(6) Using (5) in (2)

$$\vec{B}(\vec{r},t) = -\frac{1}{c^2}\,\frac{\hat{r}}{r} \times \int d^3r' \left\{\frac{\partial \vec{J}(\vec{r}',t)}{\partial t}\right\}_{t=t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}} + O\left(\frac{1}{r^2}\right)$$

$$\vec{E}(\vec{r},t) = \frac{1}{c}\,\frac{\hat{r}}{r}\int d^3r' \left\{\frac{\partial \rho(\vec{r}',t)}{\partial t}\right\}_{t=t_r} - \frac{1}{c^2}\,\frac{1}{r}\int d^3r' \underbrace{\frac{\partial t_r}{\partial t}}_{=1} \left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r} + O\left(\frac{1}{r^2}\right)$$

(7)
$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{J}(\vec{r},t) = 0$$
← usual local charge conservation condition.

$$\left\{\frac{\partial \rho(\vec{r}',t)}{\partial t}\right\}_{t=t_r} = -\left\{\vec{\nabla}_{r'}\cdot\vec{J}(\vec{r}',t)\right\}_{t=t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}}$$

acts on explicit r' and on r' dependence in t

$$= -\left\{\vec{\nabla}_{r'}\cdot\vec{J}(\vec{r}',t)\right\}_{t=t_r} - \left[\vec{\nabla}_{r'}\, t_r\right]\cdot\left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r}$$

acts only on explicit r' and not on r' dep. in t.

$$= -\left\{\vec{\nabla}_{r'}\cdot\vec{J}(\vec{r}',t)\right\}_{t=t_r} + \frac{\hat{r}}{c}\cdot\left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r} + O\left(\frac{1}{r}\right)$$

(8)
$$\int d^3r' \left\{\frac{\partial \rho(\vec{r}',t)}{\partial t}\right\}_{t=t_r} = -\underbrace{\int d^3r' \left\{\vec{\nabla}_{r'}\cdot\vec{J}(\vec{r}',t)\right\}_{t=t_r}}_{=0} + \frac{\hat{r}}{c}\cdot\int d^3r' \left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r} + O\left(\frac{1}{r}\right)$$

(∵ $\vec{J}$ is confined to finite volume)

$$= +\frac{\hat{r}}{c}\cdot\int d^3r' \left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r} + O\left(\frac{1}{r}\right)$$

⑨ Using ⑧ in ⑥

$$\vec{E}(\vec{r},t) = \frac{1}{c}\frac{\hat{r}}{r}\int d^3r' \left\{\frac{\partial}{\partial t}\rho(\vec{r}',t)\right\}_{t=t_r} - \frac{1}{c^2}\frac{1}{r}\int d^3r' \left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r} + O\left(\frac{1}{r^2}\right)$$

$$= +\frac{1}{c^2}\frac{\hat{r}}{r}\int d^3r' \; \hat{r}\cdot\left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r} - \frac{1}{c^2}\frac{1}{r}\int d^3r' \left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r} + O\left(\frac{1}{r^2}\right)$$

$$= +\frac{1}{c^2}\frac{1}{r}\int d^3r' \left[\hat{r}\left(\hat{r}\cdot\left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r}\right) - \left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t_r}\right] + O\left(\frac{1}{r^2}\right)$$

$$= \frac{1}{c^2}\frac{1}{r}\int d^3r' \; \hat{r}\times\left(\hat{r}\times\left\{\frac{\partial\vec{J}(\vec{r}',t)}{\partial t}\right\}_{t=t_r}\right) + O\left(\frac{1}{r^2}\right)$$

(using $\hat{r}\times(\hat{r}\times\vec{V}) = \hat{r}(\hat{r}\cdot\vec{V}) - \vec{V}$)

$$= -\hat{r}\times\left[-\frac{1}{c^2}\frac{\hat{r}}{r}\times\int d^3r' \left\{\frac{\partial\vec{J}(\vec{r}',t)}{\partial t}\right\}_{t=t_r}\right] + O\left(\frac{1}{r^2}\right)$$

(using ⑥)

$$= -\hat{r}\times\vec{B}(\vec{r},t) + O\left(\frac{1}{r^2}\right)$$

⑩ Thus for radiation fields we have

(i) $\vec{E} = -\hat{r}\times\vec{B} + O\left(\frac{1}{r^2}\right)$ & $\vec{B} = \hat{r}\times\vec{E} + O\left(\frac{1}{r^2}\right)$

(ii) $E^2 = B^2 + O\left(\frac{1}{r^3}\right)$

III Radiation field due to a point charge

① We saw

$$\vec{B}(\vec{r},t) = -\frac{1}{c^2}\,\frac{\hat{r}}{r} \times \int d^3r' \left\{ \frac{\partial}{\partial t} \vec{J}(\vec{r}',t) \right\}_{t = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}} + O\left(\frac{1}{r^2}\right)$$

$$\vec{E}(\vec{r},t) = -\hat{r} \times \vec{B}(\vec{r},t) + O\left(\frac{1}{r^2}\right)$$

② For a point charge

$$\rho(\vec{r},t) = q\,\delta^{(3)}(\vec{r} - \vec{r}_q(t))$$

$$\vec{J}(\vec{r},t) = q\,\vec{V}_q(t)\,\delta^{(3)}(\vec{r} - \vec{r}_q(t))$$

③ $$\vec{B}(\vec{r},t) = -\frac{q}{c^2}\,\frac{\hat{r}}{r} \times \int d^3r' \left\{ \frac{\partial}{\partial t} \left[\vec{V}_q(t)\,\delta^{(3)}(\vec{r}' - \vec{r}_q(t)) \right] \right\}_{t = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}} + O\left(\frac{1}{r^2}\right)$$

CAUTION : $\frac{\partial}{\partial t}$ does not come out of the r' integral because it is evaluated at a point where it has r' dependence.

⑦ Using ⑥ in ⑤

$$\int d^3r' \left\{ \frac{\partial}{\partial t} \left[\vec{V}_q(t)\ \delta^{(3)}(\vec{r}' - \vec{r}_q(t)) \right] \right\}_{t = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}}$$

$$= \left[\vec{a}_q(t) + \vec{a}_q(t) \left(\hat{r} \cdot \frac{\vec{V}_q(t)}{c} \right) + \frac{\vec{V}_q(t)}{c} \left(\hat{r} \cdot \vec{a}_q(t) \right) \right]_{t = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}_q(t)}{c}}$$

⑧ Using ⑦ in ③

$$\vec{B}(\vec{r},t) = -\frac{q}{c^2} \frac{\hat{r}}{r} \times \left[\vec{a}_q(t) + \vec{a}_q(t) \left(\hat{r} \cdot \frac{\vec{V}_q(t)}{c} \right) + \frac{\vec{V}_q(t)}{c} \left(\hat{r} \cdot \vec{a}_q(t) \right) \right]_{t = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}_q(t)}{c}} + O\left(\frac{1}{r^2}\right)$$

$$\vec{E}(\vec{r},t) = -\hat{r} \times \vec{B}(\vec{r},t) + O\left(\frac{1}{r^2}\right)$$

⑨ For a non-relativistic particle $\left(\frac{v}{c} << 1 \right)$

$$\vec{B}(\vec{r},t) = -\frac{q}{c^2} \frac{\hat{r}}{r} \times \vec{a}_q(t) \Big|_{t = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}_q(t)}{c}} + O\left(\frac{1}{r^2}\right)$$

$$\vec{E}(\vec{r},t) = -\hat{r} \times \vec{B}(\vec{r},t) + O\left(\frac{1}{r^2}\right)$$

IV Radiation field due to a non-relativistic point particle

1. We saw

$$\vec{B}(\vec{r},t) = -\frac{q}{c^2}\,\frac{\hat{r}}{r}\times\int d^3r'\left\{\frac{\partial}{\partial t}\left[\vec{V}_q(t)\,\delta^{(3)}(\vec{r}'-\vec{r}_q(t))\right]\right\}_{t=t-\frac{r}{c}-\frac{\hat{r}\cdot\vec{r}'}{c}} + O\left(\frac{1}{r^2}\right)$$

$$\vec{E}(\vec{r},t) = -\hat{r}\times\vec{B}(\vec{r},t)$$

2. For $\frac{v}{c} << 1$

$$t_r = t - \frac{r}{c} - \frac{\hat{r}\cdot\vec{r}'}{c}$$

$$= t - \frac{r}{c} - \hat{r}\cdot\frac{\vec{V}'}{c}\,T \longrightarrow \text{a time involved in the process}$$

$$\simeq t - \frac{r}{c} \qquad \left(\text{for } \frac{v}{c} << 1\right)$$

3.
$$\vec{B}(\vec{r},t) = -\frac{q}{c^2}\,\frac{\hat{r}}{r}\times\int d^3r'\left\{\frac{\partial}{\partial t}\left[\vec{V}_q(t)\,\delta^{(3)}(\vec{r}'-\vec{r}_q(t))\right]\right\}_{t=t-\frac{r}{c}} + O\left(\frac{1}{r^2},\frac{v}{c}\right)$$

$$= -\frac{q}{c^2}\,\frac{\hat{r}}{r}\times\left\{\frac{\partial}{\partial t}\vec{V}_q(t)\int d^3r'\,\delta^{(3)}(\vec{r}'-\vec{r}_q(t))\right\}_{t=t-\frac{r}{c}} + O\left(\frac{1}{r^2},\frac{v}{c}\right)$$

$$= -\frac{q}{c^2}\,\frac{\hat{r}}{r}\times\vec{a}_q(t)\Big|_{t=t-\frac{r}{c}} + O\left(\frac{1}{r^2},\frac{v}{c}\right)$$

$$\vec{E}(\vec{r},t) = -\hat{r}\times B(\vec{r},t) + O\left(\frac{1}{r^2},\frac{v}{c}\right)$$