Electromagnetism # ⑮

I Conservation of energy

① To derive the equation for the conservation of energy, let us introduce magnetic monopoles into the Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_e \qquad \vec{\nabla} \cdot \vec{B} = 4\pi \rho_m$$

$$-\vec{\nabla} \times \vec{E} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \frac{4\pi}{c}\vec{J}_m \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}_e$$

and force on a particle with electric charge and magnetic charge on it is

$$\vec{F} = q_e \vec{E} + \frac{1}{c} q_e \vec{v} \times \vec{B} + q_m \vec{B} - \frac{1}{c} q_m (\vec{v} \times \vec{E})$$

② It can be verified that

$$\frac{\partial \rho_e}{\partial t} + \vec{\nabla} \cdot \vec{J}_e = 0$$

$$\frac{\partial \rho_m}{\partial t} + \vec{\nabla} \cdot \vec{J}_m = 0$$

③ $\left.\begin{array}{l}\text{Rate at which}\\ \text{current does work}\\ \text{on the particle}\end{array}\right\}$ $= \vec{F} \cdot \vec{v}$

Power taken in by the particles

$$= q_e \vec{v} \cdot \vec{E} + q_m \vec{v} \cdot \vec{B}$$

$$= \vec{J}_e \cdot \vec{E} + \vec{J}_m \cdot \vec{B}$$

(4)
$$\vec{J}_e \cdot \vec{E} + \vec{J}_m \cdot \vec{B} = \frac{c}{4\pi}\left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right)\cdot\vec{E} + \frac{c}{4\pi}\left(-\vec{\nabla}\times\vec{E} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right)\cdot\vec{B}$$
$$= -\frac{1}{4\pi}\frac{c}{c}\left[\frac{\partial \vec{E}}{\partial t}\cdot\vec{E} + \frac{\partial \vec{B}}{\partial t}\cdot\vec{B}\right] + \frac{c}{4\pi}\left[(\vec{\nabla}\times\vec{B})\cdot\vec{E} - (\vec{\nabla}\times\vec{E})\cdot\vec{B}\right]$$
$$= -\frac{1}{4\pi}\frac{c}{c}\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{2}\right) - \frac{c}{4\pi}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

(5)
$$-\left(\vec{J}_e\cdot\vec{E} + \vec{J}_m\cdot\vec{B}\right) = +\frac{1}{4\pi}\frac{c}{c}\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{2}\right) + \frac{c}{4\pi}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$U \equiv \frac{1}{8\pi}(E^2+B^2) \longrightarrow \text{energy density}$$

$$\vec{S} \equiv \frac{c}{4\pi}(\vec{E}\times\vec{B}) \longrightarrow \text{energy flux density (Poynting vector)}$$

(6)
$$\frac{\partial U}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\left(\vec{J}_e\cdot\vec{E} + \vec{J}_m\cdot\vec{B}\right)$$

$$\frac{d}{dt}\int_V d^3r\, U + \oint_S d\vec{a}\cdot\vec{S} = -\int_V d^3r\left(\vec{J}_e\cdot\vec{E} + \vec{J}_m\cdot\vec{B}\right)$$

Rate of change of electromagnetic field energy with in the volume

rate of flow of electromagnetic energy out of the volume

rate of transfer of electromagnetic energy from the particles.

II Angular distribution of radiated power

① From the definition of Poynting's vector we can conclude

$$P = \oint d\vec{a} \cdot \vec{S}$$

$$= r^2 \int d\Omega \; \hat{r} \cdot \left[\frac{c}{4\pi} (\vec{E} \times \vec{B}) \right]$$

$$t_r = t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}'}{c}$$

$$= \frac{c}{4\pi} r^2 \int d\Omega \; (\hat{r} \times \vec{E}) \cdot \vec{B}$$

$$= \frac{c}{4\pi} r^2 \int d\Omega \; B^2$$

$$= \frac{c}{4\pi} r^2 \int d\Omega \left[-\frac{1}{c^2} \frac{\hat{r}}{r} \times \int d^3r' \left\{ \frac{\partial}{\partial t} \vec{J}(\vec{r}', t) \right\}_{t=t_r} \right]^2$$

$$= \frac{1}{4\pi c^3} \int d\Omega \left[\hat{r} \times \int d^3r' \left\{ \frac{\partial \vec{J}(\vec{r}', t)}{\partial t} \right\}_{t=t_1} \right]^2$$

② $$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left[\hat{r} \times \int d^3r' \left\{ \frac{\partial \vec{J}(\vec{r}', t)}{\partial t} \right\}_{t=t_1} \right]^2$$

III Power radiated by non-relativistic charged particle

(1) $$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left[\hat{r} \times \int d^3r' \left\{ \frac{\partial \vec{J}(\vec{r},t)}{\partial t} \right\}_{t=t_1} \right]^2$$

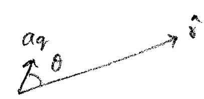
(2) $$\vec{J}(\vec{r}',t) = q\, \vec{V}_q(t)\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))$$

(3) $$\left\{\frac{\partial}{\partial t} J(\vec{r}',t)\right\}_{t=t_1} = \left\{\frac{\partial}{\partial t}\left[q\, \vec{V}_q(t)\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\right]\right\}_{t-t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}}$$

$$= \left\{\frac{\partial}{\partial t}\left[q\, \vec{V}_q(t)\, \delta^{(3)}(\vec{r}' - \vec{r}_q(t))\right]\right\}_{t=t-\frac{r}{c}} + O\left(\frac{v}{c}\right)$$

(non-relativistic approx)

(4) $$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}\left[\hat{r} \times \vec{a}_q(t_r)\right]^2 \qquad t_r = t - \frac{r}{c}$$



$$= \frac{q^2}{4\pi c^3}\left[a_q^2 - (\hat{r}\cdot\vec{a}_q)\right]_{t=t_r}$$

$$= \frac{q^2}{4\pi c^3}\left[a_q(t_r)\right]^2 \sin^2\theta$$

$$\int d\Omega\, \sin^2\theta = \frac{8\pi}{3}$$

(5) $$P = \frac{q^2}{4\pi c^3}\left[a_q\left(t-\frac{r}{c}\right)\right]^2 \int d\Omega\, \sin^2\theta$$

$$= \frac{2}{3}\frac{q^2}{c^3}\left[a_q\left(t-\frac{r}{c}\right)\right]^2 \quad \rightarrow \text{Larmor's formula.}$$

III Non-relativistic dipole radiation

① Remember (meeting 10) for a point electric dipole moment we had

$$\vec{P}(\vec{r},t) = \vec{P}_0(t)\,\delta^{(3)}(\vec{r}-\vec{r}_q(t))$$

$$\rho_{eff}(\vec{r},t) = -\vec{\nabla}\cdot\vec{P}$$

$$\vec{J}_{eff}(\vec{r},t) = \left[\frac{d}{dt}\vec{P}_0(t)\right]\delta^{(3)}(\vec{r}-\vec{r}_q(t)) \qquad t_r = t-\frac{r}{c}$$

② $$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left[\hat{r}\times\int d^3r'\left\{\frac{\partial}{\partial t}\vec{J}(\vec{r}',t)\right\}_{t=t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c}}\right]^2 + O\left(\frac{1}{r^3}\right)$$

$$= \frac{1}{4\pi c^3}\left[\hat{r}\times\left\{\frac{\partial}{\partial t}\int d^3r'\,\vec{J}(\vec{r}',t)\right\}_{t=t-\frac{r}{c}}\right]^2 + O\left(\frac{1}{r^3},\frac{v}{c}\right)$$

(non-relativistic approx)

$$= \frac{1}{4\pi c^3}\left[\hat{r}\times\left\{\frac{d^2\vec{P}_0(t)}{dt^2}\right\}_{t=t-\frac{r}{c}}\right]^2$$

③ $$P = \frac{2}{3}\frac{1}{c^3}\left\{\frac{d^2\vec{P}_0(t)}{dt^2}\right\}^2_{t=t-\frac{r}{c}}$$