Electromagnetism ⑯

I/ Maxwell's equations

① $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ — (i) $\qquad \vec{\nabla}\cdot\vec{B} = 0$ — (iii)

$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ — (ii) $\qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$ — (iv)

② (i) & (iii) can be derived from (iv) & (ii).

③
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \frac{1}{c}\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J}$$

$$0 = \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} - \frac{4\pi}{c}\frac{\partial\rho}{\partial t} \qquad \text{(using continuity eqn.)}$$

$$0 = \frac{\partial}{\partial t}\left[\vec{\nabla}\cdot\vec{E} - 4\pi\rho\right]$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho + \text{const.}$$

④
$$-\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = \frac{1}{c}\vec{\nabla}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$0 = \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B})$$

$$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0 + \text{const}$$

⑤ The constants can be chosen to be zero.

II/ Electromagnetic waves in vacuum

① $\vec{\nabla}\cdot\vec{E} = 0$ $\quad$ $\vec{\nabla}\cdot\vec{B} = 0$

$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\quad$ $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

② $$-\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{c}\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}$$

$$-\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) + \nabla^2\vec{E} = \frac{1}{c}\frac{\partial}{\partial t}\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\nabla^2\vec{E}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t)$$

③ Similarly $$\nabla^2\vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t)$$

④ Any function of the form $f(\vec{k}\cdot\vec{x} - \omega t)$ satisfies

$$\nabla^2 f(\vec{k}\cdot\vec{x} - \omega t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(\vec{k}\cdot\vec{x} - \omega t)$$

$$\underbrace{|\vec{k}|}_{\text{wave number}} = \frac{2\pi}{\underbrace{\lambda}_{\text{wavelength}}} \qquad \underbrace{\omega}_{\text{angular freq.}} = 2\pi \underbrace{\nu}_{\text{frequency}}$$

⑤ Deduce that

$$\omega = \pm kc$$

(or)

$$\nu\lambda = c$$

⑥ $e^{i\vec{k}\cdot\vec{x} - i\omega t} \longrightarrow$ plane wave or monochromatic wave.

III/ Electromagnetic waves in matter (in absence of free charges)

① $\vec{\nabla}\cdot\vec{E} = 4\pi\rho - 4\pi\,\vec{\nabla}\cdot\vec{P}$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} + \frac{4\pi}{c}\frac{\partial\vec{P}}{\partial t} + \frac{4\pi}{c}\vec{\nabla}\times\vec{M}$

② $\vec{D} \equiv \vec{E} + 4\pi\vec{P}$ $\qquad$ $\rho = 0\,,\ \vec{J} = 0$

$\vec{H} \equiv \vec{B} - 4\pi\vec{M}$

③ $\vec{\nabla}\cdot\vec{D} = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$

④ Linear medium

$\vec{P} \propto \vec{E}$ $\qquad$ $\vec{M} \propto \vec{B}$

$$\Rightarrow \quad \vec{D} = \epsilon(\vec{x},t)\,\vec{E}$$

$$\vec{H} = \frac{1}{\mu(\vec{x},t)}\,\vec{B}$$

⑤ $\vec{\nabla}\cdot\left[\epsilon\,\vec{E}\right] = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\left[\frac{\vec{B}}{\mu}\right] = \frac{1}{c}\frac{\partial}{\partial t}\left[\epsilon\,\vec{E}\right]$

⑥ If ϵ & μ are space-time independent

$\vec{\nabla}\cdot\vec{E} = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$-\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{B} = \frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$

(7) $$-\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$\nabla^2\vec{E} = \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

Similarly
$$\nabla^2\vec{B} = \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

(8) Thus for a linear homogeneous medium

$$v = \frac{c}{\sqrt{\mu\epsilon}}$$

$n \equiv \sqrt{\mu\epsilon} \rightarrow$ index of refraction

$$= \frac{c}{n}$$

IV Electromagnetic waves in conductors

1. Conductors are governed (modelled) by Ohm's law

$$\vec{J} = \sigma \vec{E}$$

$\sigma \rightarrow$ conductivity

$V = IR$

$E L = j A \frac{\rho L}{A}$

$E = \rho J \qquad \sigma = \frac{1}{\rho}$

2. $\epsilon \vec{\nabla} \cdot \vec{E} = 4\pi \rho$

$-\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \frac{\vec{B}}{\mu} = \frac{1}{c} \epsilon \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \sigma \vec{E}$

3. $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

$\frac{\partial \rho}{\partial t} + \sigma \vec{\nabla} \cdot \vec{E} = 0$

$\frac{\partial \rho}{\partial t} = -4\pi \sigma \rho$

$\Rightarrow \quad \rho(t) = e^{-4\pi \sigma t} \rho(0)$

$\Rightarrow$ at equilibrium $\rho(t) = 0$.

(ie) no free charges in a conductor.

4. After equilibrium is achieved we have.

$\epsilon \vec{\nabla} \cdot \vec{E} = 0$

$-\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \frac{\vec{B}}{\mu} = \frac{\epsilon}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \sigma \vec{E}$

5 $$-\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}\right] + \frac{1}{c}\frac{\partial}{\partial t}(\mu\sigma\vec{E})$$

$$\nabla^2\vec{E} = \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \frac{4\pi}{c^2}\mu\sigma\frac{\partial\vec{E}}{\partial t}$$

6 $$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \frac{\mu\epsilon}{c^2}\frac{\partial}{\partial t}\left[-\frac{\partial\vec{B}}{\partial t}\right] + \frac{4\pi}{c}\mu\sigma\left[-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right]$$

$$\nabla^2\vec{B} = \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} + \frac{4\pi}{c^2}\mu\sigma\frac{\partial\vec{E}}{\partial t}$$

7 Multiply by $e^{i(\tilde{k}x-\omega t)}$ and integrate.

$$-\tilde{k}^2 = -\frac{\mu\epsilon}{c^2}\omega^2 - i\frac{4\pi}{c^2}\mu\sigma\omega$$

$$\tilde{k}^2 = \mu\epsilon\frac{\omega^2}{c^2} + i\frac{4\pi}{c^2}\mu\sigma\omega$$

$$= \mu\epsilon\frac{\omega^2}{c^2}\left[1 + i\frac{4\pi\sigma}{\epsilon\omega}\right]$$

$$\tilde{k} = \frac{\sqrt{\mu\epsilon}}{\epsilon}\omega\left[1 + i\frac{4\pi\sigma}{\epsilon\omega}\right]^{1/2}$$

8 $$(1+ix)^{\frac{1}{2}} = A + iB$$

$$1 + ix = A^2 - B^2 + 2iAB$$

$$A^2 - B^2 = 1 \qquad 2AB = x$$

(9) $B = \frac{x}{2A}$

$$A^2 - \left(\frac{x}{2A}\right)^2 = 1$$

$$A^4 - A^2 - \frac{x^2}{4} = 0$$

$$A^2 = \frac{1 \pm \sqrt{1 + x^2}}{2}$$

$$A = \pm \frac{1}{\sqrt{2}} \left[1 \pm \sqrt{1+x^2}\right]^{1/2}$$

$$B = \frac{x}{2A} = \pm \frac{x}{\sqrt{2}} \frac{\left[1 \mp \sqrt{1+x^2}\right]^{1/2}}{\left[1 - 1 - x^2\right]^{1/2}}$$

imaginary

$A = \frac{x}{2B}$

$$\left(\frac{x}{2B}\right)^2 - B^2 = 1$$

$$B^4 + B^2 - \frac{x^2}{4} = 0$$

$$B^2 = \frac{-1 \pm \sqrt{1+x^2}}{2}$$

$$B = \pm \frac{1}{\sqrt{2}} \left[\sqrt{1+x^2} \pm 1\right]^{1/2}.$$

$$A = \frac{x}{2B} = \pm \frac{x}{\sqrt{2}} \frac{\left[\sqrt{1+x^2} \mp 1\right]^{1/2}}{x} = \pm \frac{1}{\sqrt{2}} \left[\sqrt{1+x^2} \mp 1\right]^{1/2}.$$

(10)

$$A = \frac{1}{\sqrt{2}} \left[\sqrt{1+x^2} + 1\right]^{1/2}$$

$$B = \frac{1}{\sqrt{2}} \left[\sqrt{1+x^2} - 1\right]^{1/2}$$

$x = \frac{4\pi\sigma}{\epsilon\omega}$

$$k = \frac{\sqrt{\mu\epsilon}}{c} \omega \frac{1}{\sqrt{2}} \left[\sqrt{1+x^2} + 1\right]$$

$$\kappa = \frac{\sqrt{\mu\epsilon}}{c} \omega \frac{1}{\sqrt{2}} \left[\sqrt{1+x^2} - 1\right]$$

(11) $\tilde{k} = k + i\kappa$

(12) $e^{i\tilde{k}x - i\omega t} = e^{ikx - i\omega t} e^{-\kappa x}$ → Attenuated wave

$d \equiv \frac{1}{\kappa}$ → skin depth