Electromagnetism HW # 1

Notation :

$$F^{\mu\nu} \equiv \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & +B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

$A^\mu \equiv (\phi, \vec{A}) \rightarrow$ 4-vector potential

$J^\mu \equiv (c\rho, \vec{J}) \rightarrow$ 4-current source

1 (a) Show that the two homogeneous (source-less) Maxwell's eqns can be written as

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

(b) Show that the two inhomogeneous Maxwell's equations can be written as

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

2 (a) Show that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

satisfies the homogeneous Maxwell's equations $(\partial_\mu \tilde{F}^{\mu\nu} = 0)$ by the property $\partial^\mu \partial^\nu = \partial^\nu \partial^\mu$.

(b) Show that the components of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ reproduce

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

3 (a) Show that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is invariant under the gauge transformation

$$A'^\mu = A^\mu - \partial^\mu \lambda .$$

(b) Show that the spatial and time components of the gauge transformation in (a) reproduce the gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda$$

$$\phi' = \phi - \frac{1}{c} \frac{\partial}{\partial t} \lambda$$

4 Show that Maxwell's equations in terms of A^μ takes the form

$$-\partial_\mu \partial^\mu A^\alpha + \partial^\alpha \partial_\mu A^\mu = \frac{4\pi}{c} J^\alpha .$$

5 (a) Show that Lorenz gauge in 4-vector notation takes the form $\partial_\mu A^\mu = 0$.

(b) Show that Maxwell's equations in the Lorenz gauge takes the form

$$-\partial_\mu \partial^\mu A^\alpha = \frac{4\pi}{c} J^\alpha .$$

6 Prove that

(a) $\delta(kx) = \frac{1}{k}\,\delta(x)$ for $k > 0$.

(Hint : Substitute $kx = y$ in the integral $\int_{-\infty}^{+\infty} dx\, f(x)\,\delta(kx)$)

(b) $\delta(x^2 - a^2) = \frac{1}{2a}\left[\,\delta(x+a) - \delta(x-a)\,\right]$ for $a > 0$.

(c) $$\delta\big(g(x)\big) = \sum_n \frac{\delta(x - a_n)}{\left\{\frac{d\,g(x)}{dx}\right\}_{x = a_n}}$$

where a_n are the roots of the equation $g(x) = 0$ and n runs over the number of roots of $g(x) = 0$. Assume that $g(x) = 0$ has distinct roots.

and that $g(x)$ is invertible.