Solution to problem ⑤ in H.W # ⑪ (ref: prob. 6.7 in Griffiths)

① $\vec{M}(\vec{r}) = M_0 \hat{z}\ \theta(a-s)$ $\quad s = \sqrt{x^2+y^2}$

② $$\vec{J}_{eff}(\vec{r}) = \vec{\nabla}\times\vec{M}(\vec{r})$$
$$= \epsilon^{ijk}\nabla^j\left[M_0\,\delta^{k3}\,\theta(a-s)\right]$$
$$= \epsilon^{ijk}\delta^{k3}M_0\,\nabla^j\theta(a-s)$$
$$= \epsilon^{ijk}\delta^{k3}M_0\,\delta(a-s)\,(-1)\frac{s^j}{s}$$
$$= -\delta(a-s)\ \epsilon^{ijk}\hat{s}^j M_0\,\delta^{k3}$$
$$= -\delta(a-s)\,(\hat{s}\times\hat{z})\,M_0$$
$$= M_0\,(\hat{z}\times\hat{s})\,\delta(a-s) = M_0\,\hat{\phi}\,\delta(a-s)$$

③ $$-\nabla^2\vec{A}(\vec{r}) = \frac{4\pi}{c}\vec{J}(\vec{r})$$
$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r'\,\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$
$$= \frac{M_0}{c}\int_{-\infty}^{+\infty}dz'\int_0^{\infty}s'ds'\int_0^{2\pi}d\phi'\ \frac{(\hat{z}'\times\hat{s}')\,\delta(a-s')}{\left[s^2+s'^2-2ss'\cos(\phi-\phi')+(z-z')^2\right]^{1/2}}$$
$$= \frac{M_0}{c}a\int_{-\infty}^{+\infty}dz'\int_0^{2\pi}d\phi'\ \frac{(\hat{z}'\times\hat{s}')}{\left[s^2+a^2-2as\cos(\phi-\phi')+(z-z')^2\right]^{1/2}}$$

4 To my knowledge evaluation of $\vec{A}(\vec{r})$ in 3 involves elliptic integrals. Thus let us evaluate $\vec{B}(\vec{r})$ straight away using

$$\vec{B}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

5
$$\begin{aligned}
\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}') &= M_0 \, \hat{\phi}' \, \delta(a - s') \times \left[(\vec{s} + z\hat{z}) - (\vec{s}' + z'\hat{z}) \right] \\
&= M_0 \, \delta(a - s') \left[\hat{\phi}' \times \vec{s} - \hat{\phi}' \times \vec{s}' + z(\hat{\phi}' \times \hat{z}) - z'(\hat{\phi}' \times \hat{z}) \right] \\
&= M_0 \, \delta(a - s') \left[-\hat{z} \, s \sin(\phi - \phi') + s'\hat{z} + (z - z')\hat{s}' \right] \\
&= M_0 \, \delta(a - s') \left[(z - z')\hat{s}' + \{ s' - s\sin(\phi - \phi') \} \hat{z} \right]
\end{aligned}$$

6 Using 5 in 4

$$\begin{aligned}
\vec{B}(\vec{r}) &= \frac{1}{c}\int_0^{\infty} s'ds' \int_0^{2\pi} d\phi' \int_{-\infty}^{+\infty} dz' \frac{M_0 \, \delta(a - s') \left[(z - z')\hat{s}' + \{ s' - s\sin(\phi - \phi') \} \hat{z} \right]}{\left[s^2 + s'^2 - 2ss'\cos(\phi - \phi') + (z - z')^2 \right]^{3/2}} \\
&= \frac{M_0}{c} a \int_0^{2\pi} d\phi' \int_{-\infty}^{+\infty} dz' \frac{\left[(z - z')\cos\phi' \, \hat{i} + (z - z')\sin\phi' \, \hat{j} + \{ a - s\sin(\phi - \phi') \} \hat{z} \right]}{\left[s^2 + a^2 - 2as\cos(\phi - \phi') + (z - z')^2 \right]^{3/2}}
\end{aligned}$$

7 $$\vec{B}(\vec{r}) = \frac{M_0}{c} a \int_0^{2\pi} d\phi' \left[\cos\phi' \, \hat{i} + \sin\phi' \, \hat{j} \right] \int_{-\infty}^{+\infty} dz' \frac{(z-z')}{\left[s^2 + a^2 - 2as\cos(\phi-\phi') + (z-z')^2 \right]^{3/2}}$$

$$+ \frac{M_0}{c} a \int_0^{2\pi} d\phi' \left[a - s\sin(\phi-\phi') \right] \hat{z} \int_{-\infty}^{+\infty} dz' \frac{1}{\left[s^2 + a^2 - 2as\cos(\phi-\phi') + (z-z')^2 \right]^{3/2}}$$

8 $$\int_{-\infty}^{+\infty} dz' \frac{(z-z')}{\left[b^2 + (z-z')^2 \right]^{3/2}} = -\int_{-\pi/2}^{+\pi/2} - \frac{b\sec^2\theta \, d\theta}{b^3 \sec^3\theta} \, b\tan\theta$$

$z - z' = b\tan\theta$

$-dz' = b\sec^2\theta \, d\theta$

$1 + \tan^2\theta = \sec^2\theta$

$$= +\frac{1}{b} \int_{-\pi/2}^{+\pi/2} d\theta \, \sin\theta$$

$$= \underline{\underline{0}}$$

9 $$\int_{-\infty}^{+\infty} dz' \frac{1}{\left[b^2 + (z-z')^2 \right]^{3/2}} = -\int_{-\pi/2}^{+\pi/2} - \frac{b\sec^2\theta \, d\theta}{b^3\sec^3\theta}$$

$$= +\frac{1}{b^2} \int_{-\pi/2}^{+\pi/2} d\theta \, \cos\theta = +\frac{2}{b^2}$$

10 Using 8 and 9 in 7

$$\vec{B}(\vec{r}) = \frac{2M_0}{c} a \, \hat{z} \int_0^{2\pi} d\phi' \frac{\left[a - s\sin(\phi-\phi') \right]}{\left[s^2 + a^2 - 2as\cos(\phi-\phi') \right]}$$

(11) $$\int_0^{2\pi} d\phi' \, \sin\phi' \, \frac{1}{[1 - b\cos(\phi-\phi')]}$$

$$= \int_0^{\pi} d\phi' \, \sin\phi' \, \frac{1}{[1 - b\cos(\phi-\phi')]} + \int_{\pi}^{2\pi} d\phi' \sin\phi' \frac{1}{[1 - b\cos(\phi-\phi')]}$$

let $\phi = 0$

$$= \int_0^{\pi} \frac{\sin\phi' \, d\phi'}{[1 - b\cos\phi']} + \int_{\pi}^{2\pi} \frac{\sin\phi' \, d\phi'}{[1 - b\cos\phi']}$$

$\phi = \phi' - \pi$
$d\phi = d\phi'$

$$= \int_0^{\pi} \frac{\sin\phi' d\phi'}{[1 - b\cos\phi']} + \int_0^{\pi} \frac{\sin(\pi+\phi) \, d\phi}{[1 - b\cos(\pi+\phi)]}$$

$$= \int_0^{\pi} \frac{\sin\phi' d\phi'}{[1 - b\cos\phi']} - \int_0^{\pi} \frac{\sin\phi \, d\phi}{[1 + b\cos\phi]}$$

$\cos\phi' = t \qquad -\sin\phi \, d\phi = dt$

$$= \int_{-1}^{+1} dt \, \frac{1}{1-bt} - \int_{-1}^{+1} \frac{dt}{1+bt}$$

$$= \int_{-1}^{+1} dt \, \frac{2bt}{1-b^2t^2} = \underline{\underline{0}}$$

(12) Using (11) in (10)

$$\vec{B}(\vec{r}) = \frac{2M_0}{c} \, \hat{z} \, a^2 \int_0^{2\pi} d\phi' \frac{1}{[s^2 + a^2 - 2as\cos(\phi-\phi')]}$$

(13) Using Gradstein's table of integrals (try to evaluate this result!)

$$\int \frac{d\phi}{A + B\cos\phi} = \begin{cases} \dfrac{2}{\sqrt{A^2-B^2}} \tan^{-1}\left[\dfrac{\sqrt{A^2-B^2}\tan\frac{\phi}{2}}{A+B}\right] & \text{for } A^2 > B^2 \\[2ex] \dfrac{1}{\sqrt{B^2-A^2}} \ln\left[\dfrac{\sqrt{B^2-A^2}\tan\frac{\phi}{2} + (A+B)}{\sqrt{B^2-A^2}\tan\frac{\phi}{2} - (A+B)}\right] & \text{for } A^2 < B^2 \end{cases}$$

(14) $$I = \int_0^{2\pi} d\phi' \frac{1}{[s^2+a^2-2as\cos\phi']}$$

$$= \int_0^{\pi} d\phi' \frac{1}{[s^2+a^2-2as\cos\phi']} + \int_0^{\pi} d\phi \frac{1}{[s^2+a^2+2as\cos\phi]}$$

$s^2+a^2 = A \qquad 2as = B$

Note that $A^2 > B^2$ for both integrals.

$$= \frac{2}{\sqrt{A^2-B^2}} \tan^{-1}\left[\frac{\sqrt{A^2-B^2}}{A-B}\right] + \frac{2}{\sqrt{A^2-B^2}} \tan^{-1}\left[\frac{\sqrt{A^2-B^2}}{A+B}\right]$$

(15) $$\sqrt{A^2-B^2} = \left[(s^2+a^2)^2 - (2as)^2\right]^{1/2} = s^2+a^2$$

$$A - B = s^2+a^2-2as = (s-a)^2$$

$$A + B = s^2+a^2-2as = (s+a)^2$$

16 $$I = \frac{2}{s^2+a^2}\left[\tan^{-1}\frac{s^2+a^2}{(s-a)^2} + \tan^{-1}\frac{s^2+a^2}{(s+a)^2}\right]$$

17 Using 16 & 14 in 12

$$\vec{B}(\vec{r}) = \frac{2M_0}{c}\,\hat{z}\,\frac{2a^2}{s^2+a^2}\left[\tan^{-1}\frac{s^2+a^2}{(s-a)^2} + \tan^{-1}\frac{s^2+a^2}{(s+a)^2}\right]$$

18 CONCERN

Is this problem not equivalent to an infinite solenoid. For an infinite solenoid we know

$$\vec{B} = \begin{cases} \frac{4\pi}{c} In\,\hat{k} & s < a \\ 0 & s > a \end{cases} \qquad \vec{J}(\vec{r}) = In\,\delta(s-a)\,\hat{\phi}$$

SHOULD WE NOT GET THIS RESULT?