Electromagnetism H.W # ⑬

① From Maxwell's equations, without introducing potentials, show that $\vec{E}$ and $\vec{B}$ satisfy the inhomogeneous wave equations

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 4\pi\left(-\vec{\nabla}\rho - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{J}\right)$$

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = \frac{4\pi}{c}\vec{\nabla}\times\vec{J}.$$

(Ask for hint if you decide to give up.) $\longrightarrow$ $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$

② A particle with charge 'q' moves along the z-axis with constant speed 'v'. Its coordinates are

$$x_q(t) = 0, \quad y_q(t) = 0, \quad z_q(t) = vt.$$

(a) Show that the charge and current densities are

$$\rho(\vec{r},t) = q\,\delta(x)\,\delta(y)\,\delta(z - z_q(t))$$
$$= q\,\delta(x)\,\delta(y)\,\delta(z - vt)$$

(problem taken from J. Schwinger et. al's book. prob. 31.1)

$$\vec{J}(\vec{r},t) = \hat{z}\,q\,v\,\delta(x)\,\delta(y)\,\delta(z - vt)$$

② (b) Show that, in the Lorentz gauge, the potentials ϕ and $\vec{A}$ satisfy

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\vec{r},t) = 4\pi q\, \delta(x)\,\delta(y)\,\delta(z-vt)$$

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}(\vec{r},t) = \frac{4\pi}{c}\,\hat{z}\, q v\, \delta(x)\,\delta(y)\,\delta(z-vt)$$

② (c) Show that, the potentials in the Lorentz gauge can be written as

$$\phi(\vec{x},t) = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy' \int_{-\infty}^{+\infty} dz' \int_{-\infty}^{+\infty} dt' \; \frac{q\,\delta(x')\,\delta(y')\,\delta(z'-vt')\,\delta\!\left[t-t'-\frac{1}{c}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right]}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$\vec{A}(\vec{x},t) = \hat{z}\,\frac{v}{c}\,\phi(\vec{x},t)$$

② (d) Evaluate the integrals in ② (c) and show that

$$\phi(\vec{x},t) = \frac{q}{\sqrt{(z-vt)^2 + (x^2+y^2)\left(1-\frac{v^2}{c^2}\right)}} \quad \text{and} \quad \vec{A}(\vec{r},t) = \hat{z}\,\frac{v}{c}\,\phi(\vec{r},t).$$

(Hint: (i) $\delta[f(x)] = \sum_r \frac{\delta(x-x_r)}{\left|\left\{\frac{\partial f(x)}{\partial x}\right\}_{x=x_r}\right|}$ $\quad r \to$ roots of $f(x)$.

(ii) Causality allows only one root to contribute.)