Solution to problem 2 in 11.10 #13

(1)
$$\phi(\vec{x},t) = q \int_{-\infty}^{+\infty} dz' \int_{-\infty}^{+\infty} dt' \; \frac{\delta(z'-vt')\;\delta\left[t-t'-\frac{1}{c}\sqrt{x^2+y^2+(z-z')^2}\right]}{\sqrt{x^2+y^2+(z-z')^2}}$$

$$= q \int_{-\infty}^{+\infty} dt' \; \frac{\delta\left[t-t'-\frac{1}{c}\sqrt{x^2+y^2+(z-vt')^2}\right]}{\sqrt{x^2+y^2+(z-vt')^2}}$$

(2)
$$f(t') = t - t' - \frac{1}{c}\sqrt{x^2+y^2+(z-vt')^2}$$

$$\frac{df(t')}{dt'} = -1 - \frac{1}{c}\,\frac{1}{2}\,\frac{2(z-vt')(-v)}{\sqrt{x^2+y^2+(z-vt')^2}}$$

$$= -1 + \frac{v}{c}\,\frac{(z-vt')}{\sqrt{x^2+y^2+(z-vt')^2}}$$

$$\delta[f(x)] = \sum_r \frac{\delta(x-x_r)}{\left|\left\{\frac{\partial f}{\partial x}\right\}_{x=x_r}\right|}$$

no modulus I think!

(3) Let $t_r$ be the roots of $f(t')=0$.
Note that in general there would be two roots

(4)
$$\phi(\vec{x},t) = q\sum_{r=1}^{2} \frac{1}{\sqrt{x^2+y^2+(z-vt_r)^2}}\;\frac{1}{\left[1-\frac{v}{c}\frac{(z-vt_r)}{\sqrt{x^2+y^2+(z-vt_r)^2}}\right]}$$

check sign.

$$= q\sum_{r=1}^{2} \frac{1}{\sqrt{x^2+y^2+(z-vt_r)^2} - \frac{v}{c}(z-vt_r)}$$

$$= q\sum_{r=1}^{2} \frac{1}{c(t-t_r) - \frac{v}{c}(z-vt_r)} \qquad (\because f(t_r)=0)$$

5 Thus our problem reduces to evaluation of $t_r$.

$$f(t_r) = 0$$

$$R^2 \equiv x^2 + y^2.$$

$$t - t_r - \frac{1}{c}\sqrt{R^2 + (z - vt_r)^2} = 0$$

$$c^2(t - t_r)^2 = R^2 + (z - vt_r)^2$$

$$t^2 + t_r^2 - 2tt_r - \frac{R^2}{c^2} - \frac{z^2}{c^2} - \frac{v^2}{c^2}t_r^2 + 2z\frac{v}{c^2}t_r = 0$$

$$t_r^2\left(1 - \frac{v^2}{c^2}\right) - 2t_r\left(t - z\frac{v}{c^2}\right) + \left(t^2 - \frac{R^2}{c^2} - \frac{z^2}{c^2}\right) = 0$$

$$t_r = \frac{\left(t - z\frac{v}{c^2}\right) \pm \sqrt{\left(t - z\frac{v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)\left(t^2 - \frac{R^2}{c^2} - \frac{z^2}{c^2}\right)}}{1 - \frac{v^2}{c^2}}$$

6 $$\left(t - z\frac{v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)\left(t^2 - \frac{R^2}{c^2} - \frac{z^2}{c^2}\right)$$

$$= t^2 + z^2\frac{v^2}{c^4} - 2z\frac{v}{c^2}t - t^2 + \frac{R^2}{c^2} + \frac{z^2}{c^2} + \frac{v^2}{c^2}t^2 - \frac{v^2}{c^2}\frac{R^2}{c^2} - \frac{v^2}{c^2}\frac{z^2}{c^2}$$

$$= \frac{R^2}{c^2}\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c^2}(z - vt)^2$$

7 Using 6 in 5

$$t_r = \frac{\left(t - z\frac{v}{c^2}\right) \pm \frac{1}{c}\sqrt{R^2\left(1 - \frac{v^2}{c^2}\right) + (z - vt)^2}}{\left(1 - \frac{v^2}{c^2}\right)}$$

8 for $v = 0$

$$t_r = t \pm \frac{1}{c}\,\frac{r}{c}$$

thus conclude that only negative roots contribute for causal process.

9 Using 7, 8 and 4

$$\phi(\vec{x},t) = \frac{q}{c(t-t_r) - \frac{v}{c}(z - v t_r)}$$

where

$$t_r = \frac{\left(t - z\frac{v}{c^2}\right) - \frac{1}{c}\sqrt{\left(1-\frac{v^2}{c^2}\right)(x^2+y^2) + (z-vt)^2}}{\left(1-\frac{v^2}{c^2}\right)}$$

10 $$M \equiv \sqrt{\left(1-\frac{v^2}{c^2}\right)(x^2+y^2) + (z-vt)^2}$$

11 $$c(t-t_r) - \frac{v}{c}(z - v t_r)$$

$$= ct - \frac{c\left(t - z\frac{v}{c^2}\right) - M}{\left(1-\frac{v^2}{c^2}\right)} - \frac{v}{c}z + \frac{v^2}{c^2}\,\frac{c\left(t - z\frac{v}{c^2}\right) - M}{\left(1-\frac{v^2}{c^2}\right)}$$

$$= \frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left[ct\left(1-\frac{v^2}{c^2}\right) - c\left(t - z\frac{v}{c^2}\right) + M - \frac{v}{c}z\left(1-\frac{v^2}{c^2}\right) + \frac{v^2}{c}t - z\frac{v^3}{c^3} - \frac{v^2}{c^2}M\right]$$

$$= M$$

12 Using 11 and 10 in 9

$$\phi(\vec{x},t) = \frac{q}{\sqrt{(z-vt)^2 + (x^2+y^2)\left(1-\frac{v^2}{c^2}\right)}}$$