Meeting No. 7

# I Motion in a constant electric field

$U = mc^2 \gamma$

$\vec{p} = m\vec{v}\gamma$

① $\dfrac{dU}{dt} = q(\vec{E}\cdot\vec{v})$

$\dfrac{d\vec{p}}{dt} = q\vec{E} + q\dfrac{\vec{v}}{c}\times\vec{B}$

② $\dfrac{d}{dt}(mc^2\gamma) = q(\vec{E}\cdot\vec{v})$

$\dfrac{d}{dt}(m\vec{v}\gamma) = q\vec{E}$

③ Break the vector parallel and ⊥r to $\vec{E}$.

$\gamma^2 = \dfrac{1}{1-\beta_{\parallel}^2-\beta_{\perp}^2}$

$\dfrac{d\gamma}{dt} = \dfrac{qE}{mc}\dfrac{v_{\parallel}}{c}$ — (i)

$\dfrac{d}{dt}(v_{\parallel}\gamma) = \dfrac{qE}{m}$ — (ii)

$\dfrac{d}{dt}(v_{\perp}\gamma) = 0$ — (iii)

④ Thus we have

$\dfrac{d\gamma}{dt} = \dfrac{qE}{mc}\dfrac{v_{\parallel}}{c}$ — (i)

$\dfrac{v_{\parallel}}{c}\gamma = \dfrac{qE}{mc}t + \text{const}$ — (ii)

$\dfrac{v_{\perp}}{c}\gamma = \text{const}$ — (iii)

5 Observe that $V_\perp r = \text{const}$ doesnot imply $V_\perp = \text{const}$ because $r$ has $V_\parallel$ inside it.

6 Let us consider the case when at $t=0$ $\frac{V_{\parallel 0}}{c} = 0$. In this case the equations reduce to

$$\frac{dr}{dt} = \frac{qE}{mc}\frac{V_\parallel}{c} \quad \text{--- (i)}$$

$$\frac{V_\parallel r}{c} = \frac{qE}{mc}\, t \quad \text{--- (ii)}$$

$$\frac{V_\perp}{c} r = \frac{V_{\perp 0}}{c} r_0 \quad \text{--- (iii)}$$

7 Let $\alpha = \frac{qE}{m}$

$$\frac{dr}{dt} = \frac{\alpha}{c}\frac{V_\parallel}{c} \quad \text{--- (i)}$$

$$\frac{V_\parallel}{c} r = \frac{\alpha}{c} t \quad \text{--- (ii)}$$

$$\frac{V_\perp}{c} r = \frac{V_{\perp 0}}{c} r_0 \quad \text{--- (iii)}$$

8 Using 7-(ii) & 7-(iii)

$$\left[1 + \left(\frac{c}{\alpha t}\right)^2\right] \beta_\parallel^2 + \beta_\perp^2 = 1 \quad \text{--- (i)}$$

$$\beta_\parallel^2 + \left[1 + \frac{1}{\beta_{\perp 0}^2 r_0^2}\right] \beta_\perp^2 = 1 \quad \text{--- (ii)}$$

9 Equations 8-(i) and 8-(ii) have the solution

$$\beta_{\parallel} = \frac{\left(\frac{\alpha t}{c}\right)}{\sqrt{1+\beta_{\perp 0}^2 r_0^2 + \left(\frac{\alpha t}{c}\right)^2}}$$

Caution:
$1+\beta_{\perp 0}^2 r_0^2 \neq r_0^2$.

$$\beta_{\perp} = \frac{\beta_{\perp 0}\, r_0}{\sqrt{1+\beta_{\perp 0}^2 r_0^2 + \left(\frac{\alpha t}{c}\right)^2}}$$

10
$$x_{\parallel} - x_{\parallel}(0) = \alpha \int_0^t dt' \frac{t'}{\sqrt{1+\beta_{\perp 0}^2 r_0^2 + \left(\frac{\alpha t'}{c}\right)^2}}$$

$$1+\beta_{\perp 0}^2 r_0^2 + \left(\frac{\alpha t'}{c}\right)^2 = s \qquad \frac{\alpha^2}{c^2}\, 2\, t'\, dt' = ds$$

$$x_{\parallel}(t) - x_{\parallel}(0) = \alpha \int_{1+\beta_{\perp 0}^2 r_0^2}^{1+\beta_{\perp 0}^2 r_0^2 + \left(\frac{\alpha t}{c}\right)^2} \frac{ds}{2\frac{\alpha^2}{c^2}} \frac{1}{\sqrt{s}}$$

$$= \frac{c^2}{2\alpha}\, 2\sqrt{s}\,\Big|_{s=1+\beta_{\perp 0}^2 r_0^2}^{s=1+\beta_{\perp 0}^2 r_0^2+\left(\frac{\alpha t}{c}\right)^2}$$

$$= \frac{c^2}{\alpha}\left[\sqrt{1+\beta_{\perp 0}^2 r_0^2 + \left(\frac{\alpha t}{c}\right)^2} - \sqrt{1+\beta_{\perp 0}^2 r_0^2}\right]$$

$$= \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2 r_0^2}\left[\sqrt{1+\frac{\left(\frac{\alpha t}{c}\right)^2}{1+\beta_{\perp 0}^2 r_0^2}} - 1\right]$$

(11) At $t=0$ let $x_{\parallel}(0) = 0$. (ie) the particle is at the origin at $t=0$.

$$x_{\parallel}(t) = \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2 r_0^2}\left[\sqrt{1+\frac{\left(\frac{\alpha t}{c}\right)^2}{1+\beta_{\perp 0}^2 r_0^2}} - 1\right]$$

$$x_{\parallel}(t) + \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2 r_0^2} = \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2 r_0^2}\sqrt{1+\frac{\left(\frac{\alpha t}{c}\right)^2}{1+\beta_{\perp 0}^2 r_0^2}}$$

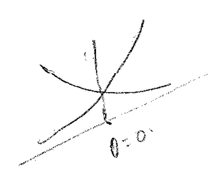
$$\left\{x_{\parallel}(t) + \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2 r_0^2}\right\}^2 = \frac{c^4}{\alpha^2}\left(1+\beta_{\perp 0}^2 r_0^2 + \left(\frac{\alpha t}{c}\right)^2\right)$$

$$\left\{x_{\parallel}(t) + \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2 r_0^2}\right\}^2 - c^2 t^2 = \frac{c^4}{\alpha^2}\left(1+\beta_{\perp 0}^2 r_0^2\right)$$

(12) Compare this with the result in meeting No. (4).

(13) Similarly using (9) we have.

$$x_{\perp}(t) - x_{\perp}(0) = c\beta_{\perp 0} r_0 \int_0^t dt' \frac{1}{\sqrt{1+\beta_{\perp 0}^2 r_0^2 + \left(\frac{\alpha t'}{c}\right)^2}}$$




$$\frac{\alpha^2 t'^2}{c^2} = \left(1+\beta_{\perp 0}^2 r_0^2\right)\sinh^2\theta$$

$$\frac{\alpha}{c}dt' = \sqrt{1+\beta_{\perp 0}^2 r_0^2}\,\cosh\theta\, d\theta$$

$\sinh\theta = 0$

$e^{\theta} - e^{-\theta} = 0$

$e^{\theta} = e^{-\theta}$

$\theta = 0$

(14)
$$x_\perp(t) - x_\perp(0) = c\,\beta_{\perp 0}\, r_0\, \frac{c}{\alpha} \int_{\theta=0}^{\theta=\sinh^{-1}\left(\frac{\frac{\alpha t}{c}}{\sqrt{1+\beta_{\perp 0}^2 r_0^2}}\right)} d\theta$$

$$= \beta_{\perp 0}\, r_0\, \frac{c^2}{\alpha} \sinh^{-1}\left(\frac{\frac{\alpha t}{c}}{\sqrt{1+\beta_{\perp 0}^2 r_0^2}}\right)$$

(15) Let at $t=0$ $x_\perp(0) = 0$.

$$x_\perp(t) = \beta_{\perp 0}\, r_0\, \frac{c^2}{\alpha} \sinh^{-1}\left(\frac{\frac{\alpha t}{c}}{\sqrt{1+\beta_{\perp 0}^2 r_0^2}}\right)$$

$$\sinh\left\{\frac{\alpha}{c^2}\,\frac{1}{\beta_{\perp 0} r_0}\, x_\perp(t)\right\} = \frac{\alpha t/c}{\sqrt{1+\beta_{\perp 0}^2 r_0^2}}$$

$$\frac{\alpha^2 t^2}{c^2} = \left(1 + \beta_{\perp 0}\, r_0^2\right) \sinh^2\left\{\frac{\alpha}{c}\,\frac{1}{\beta_{\perp 0} r_0}\, x_\perp(t)\right\}$$

(16) Using (11) and (15) we have

$$\left\{x_\parallel(t)^2 + \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2 r_0^2}\right\}^2 - c^2 t^2 = \frac{c^4}{\alpha^2}\left(1+\beta_{\perp 0}^2 r_0^2\right)$$

$$c^2 t^2 = \frac{c^4}{\alpha^2}\left(1+\beta_{\perp 0}^2 r_0^2\right) \sinh^2\left\{\frac{\alpha}{c}\,\frac{1}{\sqrt{1+\beta_{\perp 0}^2 r_0^2}}\, x(t)\, \sqrt{1+\frac{1}{\beta_{\perp 0}^2 r_0^2}}\right\}$$

17 Using 16

$$\left\{x_{\parallel}(t) + \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2\gamma_0^2}\right\}^2 = \frac{c^4}{\alpha^2}(1+\beta_{\perp 0}^2\gamma_0^2)\left[1 + \sinh^2\left\{\frac{\alpha}{c}\frac{1}{\sqrt{1+\beta_{\perp 0}^2\gamma_0^2}}\, x_{\perp}(t)\sqrt{1+\frac{1}{\beta_{\perp 0}^2\gamma_0^2}}\right\}\right]$$

$$\left\{x_{\parallel}(t) + \frac{c^2}{\alpha}\sqrt{1+\beta_{\perp 0}^2\gamma_0^2}\right\}^2 = \frac{c^4}{\alpha^2}\left(1+\beta_{\perp 0}^2\gamma_0^2\right)\cosh^2\left\{\frac{\alpha}{c}\frac{1}{\sqrt{1+\beta_{\perp 0}^2\gamma_0^2}}\, x_{\perp}(t)\sqrt{1+\frac{1}{\beta_{\perp 0}^2\gamma_0^2}}\right\}$$

18 Denote $\bar{\alpha} = \dfrac{\alpha}{\sqrt{1+\beta_{\perp 0}^2\gamma_0^2}} = \dfrac{\frac{qE}{m}}{\sqrt{1+\beta_{\perp 0}^2\gamma_0^2}}$

$$\left(x_{\parallel}(t) + \frac{c^2}{\bar{\alpha}}\right)^2 - c^2t^2 = \frac{c^4}{\bar{\alpha}^2}$$

$$c^2t^2 = \frac{c^4}{\bar{\alpha}^2}\sinh^2\left\{\frac{\bar{\alpha}}{c}\, x_{\perp}(t)\sqrt{1+\frac{1}{\beta_{\perp 0}^2\gamma_0^2}}\right\}$$

$$\left(x_{\parallel}(t) + \frac{c^2}{\bar{\alpha}}\right)^2 = \frac{c^4}{\bar{\alpha}^2}\cosh^2\left\{\frac{\bar{\alpha}}{c}\, x_{\perp}(t)\sqrt{1+\frac{1}{\beta_{\perp 0}^2\gamma_0^2}}\right\}$$

19 Take the non-relativistic limits

$$\left.\begin{array}{l} qE << mc. \\ \beta_{\perp 0} << 0 \quad \beta_{\parallel 0} << 1 \end{array}\right\}$$

THINK!!

Do it correctly.

## II Transformation of $\vec{E}$ and $\vec{B}$

① $$\bar{F}^{\mu\nu} = L^{\mu}{}_{\alpha} L^{\nu}{}_{\beta} F^{\alpha\beta}$$
$$= L^{\mu}{}_{\alpha} F^{\alpha\beta} L^{\nu}{}_{\beta}$$

② In matrix notation

$$\bar{F} = L \cdot F \cdot L^{T}$$

③ Two independent invariants of F are (see problem 3 for motivation — eigenvalues of F)

$$F^{\mu\nu} F_{\mu\nu} = 2(B^2 - E^2)$$
$$F^{\mu\nu} \tilde{F}_{\mu\nu} = 4 \vec{B} \cdot \vec{E}$$

③ Possible configurations:

(i) $B^2 - E^2 > 0$

(ii) $B^2 - E^2 < 0$

(iii) $B^2 - E^2 = 0$
(a) $\vec{B} \cdot \vec{E} = 0$
(b) $\vec{B} \cdot \vec{E} > 0$
(c) $\vec{B} \cdot \vec{E} < 0$