Homework No. (7)

(1) Use the expressions for $F^{\mu\nu}$, $F_{\mu\nu}$, $\tilde{F}^{\mu\nu}$, and $\tilde{F}_{\mu\nu}$ to evaluate the following:

(a) $F^{\mu\nu} F_{\mu\nu} = - \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} = 2(B^2 - E^2)$

(b) $F^{\mu\nu} \tilde{F}_{\mu\nu} = 4(\vec{B} \cdot \vec{E})$

(2) Prove:

(a) If $B^2 - E^2 > 0$ in a frame, then there exists a frame in which the field is purely magnetic.

(b) If $B^2 - E^2 < 0$ in a frame, then there exists a frame in which the field is purely electric.

(c) If $B^2 - E^2 = 0$ in a frame, then there exists a frame in which

(i) $\vec{B}' \perp \vec{E}'$, if $\vec{B} \cdot \vec{E} = 0$

(ii) $\vec{B}' \parallel \vec{E}'$, if $\vec{B} \cdot \vec{E} > 0$

(iii) $\vec{B}' \parallel -\vec{E}'$, if $\vec{B} \cdot \vec{E} < 0$

(3) Eigenvalues of the field tensor $F^{\mu\nu}$:

(a) Show that:

(i) $$F^{\mu\lambda}F_{\lambda\nu} \equiv \begin{bmatrix} E^2 & B_yE_z - B_zE_y & B_zE_x - B_xE_z & B_xE_y - B_yE_x \\ B_zE_y - B_yE_z & E_x^2 - B_y^2 - B_z^2 & B_xB_y + E_xE_y & B_xB_z + E_xE_z \\ B_xE_z - B_zE_x & B_xB_y + E_xE_y & E_y^2 - B_x^2 - B_z^2 & B_yB_z + E_yE_z \\ B_yE_x - B_xE_y & B_xB_z + E_xE_z & B_yB_z + E_yE_z & E_z^2 - B_x^2 - B_y^2 \end{bmatrix}$$

(ii) $$\tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu} \equiv \begin{bmatrix} B^2 & B_yE_z - B_zE_y & B_zE_x - B_xE_z & B_xE_y - B_yE_x \\ B_zE_y - B_yE_z & B_x^2 - E_y^2 - E_z^2 & B_xB_y + E_xE_y & B_xB_z + E_xE_z \\ B_xE_z - B_zE_x & B_xB_y + E_xE_y & B_y^2 - E_x^2 - E_z^2 & B_yB_z + E_yE_z \\ B_yE_x - B_xE_y & B_xB_z + E_xE_z & B_yB_z + E_yE_z & B_z^2 - E_x^2 - E_y^2 \end{bmatrix}$$

(iii) $$F^{\mu\lambda}\tilde{F}_{\lambda\nu} = \delta^\mu{}_\nu\,(\vec{E}\cdot\vec{B})$$

(iv) $$\tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu} - F^{\mu\lambda}F_{\lambda\nu} = \delta^\mu{}_\nu\,(B^2 - E^2)$$

(b) Define: $f = \frac{1}{2}(B^2 - E^2)$ and $g = (\vec{B}\cdot\vec{E})$.

Write (a)-(iii) and (a)-(iv) as matrix equations and thus arrive at the following two equations

$$F\cdot\tilde{F} = g\,\mathbb{1}$$

$$\tilde{F}\cdot\tilde{F} - F\cdot F = 2f\,\mathbb{1}$$

(c) Using the first equation in (b) write

$$\tilde{F} = g\, F^{-1}$$

and thus arrive at the following equation

$$g^2\, F^{-1} \cdot F^{-1} - F \cdot F = 2f\, \mathbb{1} .$$

(d) Using (c) conclude that the eigenvalues 'λ' of the field tensor $F^{\mu\nu}$ satisfy the equation

$$\lambda^4 + 2f\lambda^2 - g^2 = 0$$

(e) Using (d) evaluate the four eigenvalues of the field tensor $F^{\mu\nu}$ to be $\pm\lambda_1$ and $\pm\lambda_2$, where,

$$\lambda_1 = \left[-f - \sqrt{f^2+g^2}\right]^{\frac{1}{2}} = \frac{i}{\sqrt{2}}\left[\sqrt{f+ig} + \sqrt{f-ig}\right]$$

$$\lambda_2 = \left[-f + \sqrt{f^2+g^2}\right]^{\frac{1}{2}} = \frac{i}{\sqrt{2}}\left[\sqrt{f+ig} - \sqrt{f-ig}\right]$$

4 (a) Show that, if $B^2 - E^2 = 0$, then eigen values of $F^{\mu\nu}$ are $\pm\sqrt{g}$ and $\pm i\sqrt{g}$.

(b) Show that, if $\vec{B}\cdot\vec{E} = 0$, then eigen values of $F^{\mu\nu}$ are $0, 0$, and $\pm\sqrt{2f}$.

5 Prove that there is no Lorentz frame connecting two such frames such that the field is purely magnetic in origin in one and purely electric in origin in the other.

6 Equations of motion of a particle in an electromagnetic field is described by

$$\frac{dU}{dt} = q\,\vec{E}\cdot\vec{v}$$

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\,\frac{\vec{v}}{c}\times\vec{B}\,.$$

Show that for a particle in a constant electric field $\vec{E}$ which is initially at rest the above equations reduces to a 1-dimensional problem

$$\frac{d\gamma}{dt} = \frac{qE}{m}\,\frac{v}{c} \quad \text{and} \quad \frac{d}{dt}(v\gamma) = \frac{qE}{m}\,.$$

(a) Integrate the second equation to get

$$v = \frac{dx}{dt} = \frac{\frac{qE}{m}\,t}{\sqrt{1 + \left(\frac{qE}{mc}\,t\right)^2}}$$

(ref: notes in meeting no. 5).

(b) Integrate it further to get

$$\{x(t) - x(0)\}^2 - c^2 t^2 = \frac{c^4}{\left(\frac{qE}{m}\right)^2}$$

which is the equation of a hyperbola.

(c) Find the non-relativistic limit of the above equation (ref: problem 1, H.W #4).