Math 251 Worksheet 2-Section 13.3 Name________________________

K.A. Pericak-Spector

16.3 deals only with line integrals where the integrand is a vector-valued function. It is used to make your computation easier. The idea is that the integrand, **F**, is conservative. **(If it is not, this method will not work).** Since **F** is conservative, it is the gradient of some potential $F$, i.e. $\mathbf{F}= \nabla f$. Then $\int_C \mathbf{F}\cdot d\mathbf{r}= \int_C \nabla f \cdot d\mathbf{r}= f(b) - f(a)$ where $a$ and $b$ are the initial and final points of the curve. **Notice the answer is independent of path!**

Example. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}= (x^2 + y^2)\ \mathbf{i}+2xy\mathbf{j}$ $\quad \mathbf{r}= t^3\mathbf{i}+t^2\mathbf{j}$ $\quad 0 \le t \le 2$

Step 1. Identify $P$ and $Q$. Here. $P = x^2 + y^2 \qquad Q = 2xy$

Step 2. Compute $\dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial P}{\partial y}$. Here $\dfrac{\partial Q}{\partial x} = 2y \qquad \dfrac{\partial P}{\partial y} = 2y$

Step 3. Are they equal? Here, yes.

Step 4. Find $f$ such that $f_x = x^2 + y^2$ and $f_y = 2xy$.

Integrating $f_x$ with respect to $x$ treating $y$ as constant

$$f = \frac{x^3}{3} + xy^2 + g(y)$$

This is the answer if you know $g$! Differentiate with respect to $y$ treating $x$ as a constant

$$f_y = 2xy + g'(y)$$

This must be equal to the other $f_y = Q = 2xy$.

$$\begin{aligned} 2xy + g'(y) &= 2xy \\ \Rightarrow g'(y) &= 0 \end{aligned}$$

So $f = \dfrac{x^3}{3} + xy^2$.

Step 5. Check your answer. Here $\nabla f = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$

Step 6. Find initial and final point or curve. Here since $0 \le t \le 2$ $\quad \begin{array}{l} \mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j} \\ \mathbf{r}(2) = 8\mathbf{i} + 4\mathbf{j} \end{array}$

Step 7. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r} = f(8, 4) - f(0, 0) = \dfrac{8^3}{3} + 8 \cdot 4^2 - \left(\dfrac{0^3}{3} + 0 \cdot 0^2\right)$

Example: Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ $\quad \mathbf{F} = e^y\mathbf{i} + xe^y\mathbf{j} \qquad \mathbf{r} = \sin^3 t\mathbf{i} + \cos^2 t\mathbf{j} \qquad 0 \le t \le \frac{\pi}{2}$

Step 1. Identify $P$ and $Q$. Here $P = e^y,\ Q = xe^y$.

Step 2. Compute $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$. Here $\frac{\partial Q}{\partial x} = e^y, \frac{\partial P}{\partial y} = e^y$.

Step 3. Are they equal? Here, yes.

Step 4. Find $f$ such that $\begin{array}{l} f_x = e^y \\ f_y = xe^y \end{array}$

Integrate $f_x$ with respect to $x$ treating $y$ as constant.

$$f = xe^y + g(y)$$

This is the answer if you know $g$! Differentiate with respect to $y$ treating $x$ as a constant

$$f_y = xe^y + g'(y)$$

This must be equal to the other $f_y = xe^y$; $\qquad xe^y + g'(y) = xe^y \Rightarrow g'(y) = 0 \qquad f = xe^y$.

Step 5. Check your answer. Here $\nabla f = e^y\mathbf{i} + xe^y\mathbf{j}$

Step 6. Find initial and final point on curve. Here since $0 \le t \le \frac{\pi}{2} \qquad r(0) = 0\mathbf{i}+1\mathbf{j} \qquad r\left(\frac{\pi}{2}\right) = 1\mathbf{i}+0\mathbf{j}$.

Step 7. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r} - f(1,0) = f(0,1) = 1e^0 - 0e' = 1$.

Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r} \qquad \mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j} \qquad \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} \qquad 0 \le t \le 1$

Step 1. Identify $P$ and $Q$.

Step 2. Compute $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$.

Step 3. Are they equal? If yes, continue. If not, do as in previous section.

Step 4. Find $f$ such that $f_x = P$ and $f_y = Q$.

Step 5. Check your answer. i.e. Show $\mathbf{F} = \nabla f$.

Step 6. Find initial and final point on curve.

Step 7. Evaluate integral.

B. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} \qquad \mathbf{F} = (e^y + x)\mathbf{i} + (xe^y + \sin y)\mathbf{j} \qquad \mathbf{r} = e^t \cos t\mathbf{i} + e^{-t} \sin t\mathbf{j} \qquad 0 \leq t \leq \frac{\pi}{4}.$

Step 1. Identify $P$ and $Q$.

Step 2. Compute $\dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial P}{\partial y}$.

Step 3. Are they equal? If yes, continue. If not do as in previous section.

Step 4. Find $f$ such that $f_x = P$ and $f_y = Q$.

Step 5. Check your answer. i.e. Show $\mathbf{F} = \nabla f$.

Step 6. Find initial and final point on curve.

Step 7. Evaluate integral.

C. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} \qquad \mathbf{F} = y\mathbf{i} - x\mathbf{j} \qquad \mathbf{r}(t) = \tan t\mathbf{i} + \sec^2 t\mathbf{j} \qquad \frac{\pi}{6} \leq t \leq \frac{\pi}{3}$

Step 1. Identify $P$ and $Q$.

Step 2. Compute $\dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial P}{\partial y}$.

Step 3. Are they equal? If yes, continue. If not do as in previous section.

Step 4. Find $f$ such that $f_x = P$ and $f_y = Q$.

Step 5. Check your answer. i.e. Show $\mathbf{F} = \nabla f$.

Step 6. Find initial and final point on curve.

Step 7. Evaluate integral.