Solutions

Exam No. 01 (Fall 2013)

PHYS 320: Electricity and Magnetism I

Date: 2013 Sep 18

1. (**20 points.**) The relation between the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{1}$$

For a constant (homogeneous in space) magnetic field $\mathbf{B}$, verify that

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \tag{2}$$

is a possible vector potential by showing that Eq. (2) satisfies Eq. (1).

2. (**20 points.**) A gyroid is an (infinitely connected triply periodic minimal) surface discovered by Alan Schoen in 1970. Schoen presently resides in Carbondale and was a professor at SIU in the later part of his career. Apparently, a gyroid is approximately described by the surface

$$f(x, y, z) = \cos x \sin y + \cos y \sin z + \cos z \sin x \tag{3}$$

when $f(x, y, z) = 0$. Using the fact that the gradient operator

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{4}$$

determines the normal vectors on a surface, evaluate

$$\boldsymbol{\nabla} f(x, y, z). \tag{5}$$

3. (**20 points.**) Evaluate the integral

$$\int_{-1}^{1} \frac{\delta(1 - 3x)}{x}\, dx. \tag{6}$$

Hint: Be careful to avoid a possible error in sign.

4. (**20 points.**) Evaluate the vector area of a hemispherical bowl of radius R given by

$$\mathbf{a} = \int_S d\mathbf{a}. \tag{7}$$

5. (**20 points.**) Evaluate the number evaluated by the expression

$$\frac{1}{2}\left[\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right]\cdot(\rho\hat{\boldsymbol{\rho}}), \tag{8}$$

where $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are the unit vectors for cylindrical coordinates (ρ, ϕ) given by

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}, \tag{9}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}. \tag{10}$$

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(1)

$$\vec{\nabla} \times \vec{A} = \frac{1}{2} \vec{\nabla} \times (\vec{B} \times \vec{r})$$

$$(\vec{\nabla} \times \vec{A})_i = \frac{1}{2} \epsilon_{ijk} \nabla_j (\epsilon_{kmn} B_m r_n)$$

$$= \frac{1}{2} \epsilon_{kij} \epsilon_{kmn} \nabla_j (B_m r_n)$$

$$= \frac{1}{2} \epsilon_{kij} \epsilon_{kmn} B_m \nabla_j r_n \qquad (\because \vec{B} \text{ is independent of position.})$$

$$= \frac{1}{2} \epsilon_{kij} \epsilon_{kmn} B_m \delta_{jn} \qquad (\text{using } \nabla_j r_n = \delta_{jn})$$

$$= \frac{1}{2} (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) B_m \delta_{jn}$$

$$= \frac{1}{2} (B_i \delta_{jj} - B_j \delta_{ji}) \qquad (\text{using } \delta_{ii} = 3)$$

$$= \frac{1}{2} (3 B_i - B_i)$$

$$= B_i = \vec{B}.$$

(2)

$$f(x,y,z) = \cos x \sin y + \cos y \sin z + \cos z \sin x.$$

$$\vec{\nabla} f = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right] f$$

$$= \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}$$

$$= \hat{i} (-\sin x \sin y + \cos x \cos z) + \hat{j} (\cos x \cos y - \sin y \sin z)$$
$$+ \hat{k} (+ \cos y \cos z - \sin z \sin x)$$

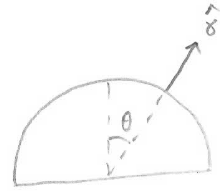
$$= -\hat{i} (\sin x \sin y - \cos x \cos z) - \hat{j} (\sin y \sin z - \cos y \cos x)$$
$$- \hat{k} (\sin z \sin x - \cos z \cos y).$$

$$\delta(1-ax) = \frac{1}{|a|}\,\delta\left(x-\frac{1}{a}\right)$$

3 $$\int_{-1}^{+1} dx \,\frac{1}{x}\,\delta(1-3x) = \int_{-1}^{+1} dx\,\frac{1}{x}\,\frac{1}{3}\,\delta\left(x-\frac{1}{3}\right)$$

$$= \frac{1}{(1/3)}\,\frac{1}{3}$$

$$= 1.$$



4 $$\vec{a} = \int_S d\vec{a}$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{2\pi} d\phi\ \hat{r}\ R^2$$

Using $$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

we have

$$\vec{a} = R^2 \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi \left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right]$$

$$= \hat{i}\,R^2 \int_0^{\pi/2} d\theta\,\sin^2\theta\, \underbrace{\int_0^{2\pi} d\phi\,\cos\phi}_{=0} + \hat{j}\,R^2 \int_0^{\pi/2} d\theta\,\sin^2\theta\, \underbrace{\int_0^{2\pi} d\phi\,\sin\phi}_{=0}$$

$$+ \hat{k}\,R^2 \int_0^{\pi/2} d\theta\,\sin\theta\cos\theta\, \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\cos\theta = t$$
$$-\sin\theta\, d\theta = dt$$

$$= 2\pi R^2\,\hat{k} \int_0^{\pi/2} \sin\theta\, d\theta\,\cos\theta$$

$$= 2\pi R^2\,\hat{k} \int_1^0 (-1)\, dt\ t$$

$$= 2\pi R^2\,\hat{k}\ \frac{t^2}{2}(-1)\Big|_{t=1}^{t=0}$$

$$= \pi R^2\,\hat{k}$$

5 $\hat{\rho} = \cos\phi \, \hat{i} + \sin\phi \, \hat{j}$

$\hat{\phi} = -\sin\phi \, \hat{i} + \cos\phi \, \hat{j}$.

Observe that

$$\frac{\partial}{\partial \rho} \hat{\rho} = 0 \quad \text{—— (i)}$$

$$\frac{\partial}{\partial \phi} \hat{\rho} = -\sin\phi \, \hat{i} + \cos\phi \, \hat{j} \quad \text{—— (ii)}$$

$$= \hat{\phi} .$$

So,

$$\frac{1}{2} \left[\hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} \right] \cdot (\rho \, \hat{\rho})$$

$$= \frac{1}{2} \hat{\rho} \cdot \frac{\partial}{\partial \rho} (\rho \, \hat{\rho}) + \frac{1}{2} \hat{\phi} \cdot \frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho \, \hat{\rho})$$

$$= \frac{1}{2} \hat{\rho} \cdot \Big[\hat{\rho} + \rho \underbrace{\frac{\partial \hat{\rho}}{\partial \rho}}_{\substack{= 0 \\ \text{using (i)}}} \Big] + \frac{1}{2} \hat{\phi} \cdot \frac{1}{\rho} \Big[\underbrace{\left(\frac{\partial \rho}{\partial \phi} \right)}_{= 0} \hat{\rho} + \rho \underbrace{\frac{\partial \hat{\rho}}{\partial \phi}}_{\substack{\hat{\phi} \\ \text{using (ii)}}} \Big]$$

$$= \frac{1}{2} \hat{\rho} \cdot \hat{\rho} + \frac{1}{2} \hat{\phi} \cdot \hat{\phi}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1 .$$