Solutions

# Exam No. 02 (Fall 2013)

## PHYS 320: Electricity and Magnetism I

Date: 2013 Oct 23

1. (**20 points.**) The electric field due to a point dipole $\mathbf{d}$ at a distance $\mathbf{r}$ away from dipole is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{d}\big]. \tag{1}$$

Consider the case when the point dipole is positioned at the origin and is pointing in the $z$-direction, i.e., $\mathbf{d} = d\,\hat{\mathbf{z}}$.

(a) Qualitatively plot the electric field lines for the dipole $\mathbf{d}$. (Hint: You do not have to depend on Eq. (1) for this purpose. An intuitive knowledge of electric field lines should be the guide.)

(b) Find the (simplified) expression for the electric field on the positive $z$-axis. (Hint: On the positive $z$-axis we have, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ and $r = z$.)

2. (**30 points.**) Consider a solid sphere of radius $R$ with total charge $Q$ distributed inside the sphere with a charge density

$$\rho(\mathbf{r}) = br^2\,\theta(R - r), \tag{2}$$

where $r$ is the distance from the center of sphere, and $\theta(x) = 1$, if $x > 0$, and 0 otherwise.

(a) Integrating the charge density over all space gives you the total charge $Q$. Thus, determine the constant $b$ in terms of $Q$ and $R$.

(b) Using Gauss's law find the electric field inside and outside the sphere.

(c) Plot the electric field as a function of $r$.

3. (**20 points.**) In class we evaluated the electric potential due to a solid sphere with uniform charge density $Q$. The angular integral in this evaluation involved the integral

$$\frac{1}{2}\int_{-1}^{1} dt \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}. \tag{3}$$

Evaluate the integral for $r < r'$ and $r' < r$, where $r$ and $r'$ are distances measured from the center of the sphere. (Hint: Substitute $r^2 + r'^2 - 2rr't = y$.)

4. (**30 points.**) The charge density for a point charge $q_a$ is described by

$$\rho(\mathbf{r}) = q_a \delta^{(3)}(\mathbf{r} - \mathbf{r}_a), \tag{4}$$

where $\mathbf{r}_a$ is the position of the charge.

(a) Evaluate the electric potential due to the point charge using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{5}$$

(Hint: Use the $\delta$-function property to evaluate the integrals.)

(b) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (5),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{6}$$

(c) Evaluate the force exerted by the charge $q_a$ on another charge $q_b$, at position $\mathbf{r}_b$, using the expression for electric field you obtained using Eq. (6) in

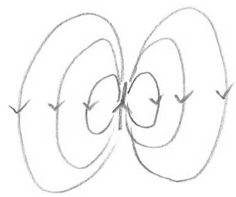
$$\mathbf{F} = q_b \mathbf{E}(\mathbf{r}_b). \tag{7}$$

To provide a check for your calculation, the answer for the expression for the force is provided here:

$$\mathbf{F} = \frac{q_a q_b}{4\pi\varepsilon_0} \frac{\mathbf{r}_b - \mathbf{r}_a}{|\mathbf{r}_b - \mathbf{r}_a|^3}. \tag{8}$$

Prob 1, Exam-2,

(a)



(b)

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3d(\hat{z}\cdot\hat{r})\hat{r} - d\hat{z}\right] \qquad (\because \hat{r} = \hat{z})$$

$$= \frac{d}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\hat{z}\cdot\hat{z})\hat{z} - \hat{z}\right]$$

$$= \frac{d}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\hat{z} - \hat{z}\right]$$

$$= \frac{2d}{4\pi\epsilon_0}\frac{1}{r^3}\hat{z}.$$

Prob 2, Exam-2

(a)

$$Q = \int d^3r\, \rho(\vec{r})$$

$$= \int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta \;\; b r^2\, \theta(R-r)$$

$$= 4\pi b \int_0^R r^4 dr$$

$$= 4\pi b \frac{R^5}{5}$$

$$b = \frac{Q}{\left(\frac{4\pi}{5}R^5\right)} = \frac{5Q}{4\pi R^5}$$

(b) $\underline{r > R}$ :

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{en}.$$

$$E \, 4\pi r^2 = \frac{1}{\epsilon_0} Q$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r}$$

$\underline{r < R}$.

$$\frac{Q_{en}}{Q} = \frac{4\pi b \frac{r^5}{5}}{4\pi b \frac{R^5}{5}} = \frac{r^5}{R^5}$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{en}$$

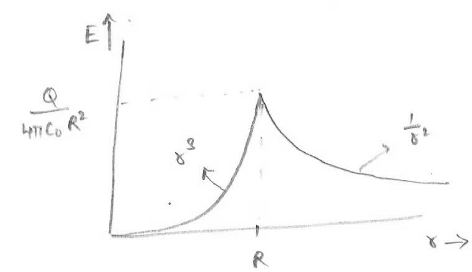
$$E \left(4\pi r^2\right) = \frac{Q}{\epsilon_0} \frac{r^5}{R^5}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{r^3}{R^5} \hat{r}$$

Thus,

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon_0} \dfrac{r^3}{R^5} \hat{r} & r < R \\[2ex] \dfrac{Q}{4\pi\epsilon_0} \dfrac{1}{r^2} \hat{r} & r > R \end{cases}$$

(c)



Prob 3, Exam-2

$$I = \frac{1}{2}\int_{-1}^{1} dt \, \frac{1}{\sqrt{r^2+r'^2-2rr't}}$$

$$r^2+r'^2-2rr't = y$$

$$-2rr'\,dt = dy$$

$$I = \frac{1}{2}\int_{(r+r')^2}^{(r-r')^2} -\frac{dy}{2rr'}\,\frac{1}{\sqrt{y}}$$

$$= \frac{1}{4rr'}\int_{(r-r')^2}^{(r+r')^2} \frac{dy}{\sqrt{y}}$$

$$\frac{y^{-\frac{1}{2}+1}}{\left(\frac{1}{2}\right)} = 2\sqrt{y}$$

$$= \frac{1}{4rr'}\, 2\sqrt{y}\,\Big|_{y=(r-r')^2}^{y=(r+r')^2}$$

$$= \frac{1}{2rr'}\left[(r+r') - |r-r'|\right]$$

$$= \begin{cases} \frac{1}{2rr'}\left[(r+r') - (r-r')\right] & r > r' \\ \frac{1}{2rr'}\left[(r+r') - (r'-r)\right] & r < r' \end{cases}$$

$$= \begin{cases} \frac{r'}{r} & r > r' \\ \frac{r}{r'} & r < r' \end{cases}$$

$$r_< = \mathrm{Min}\,(r, r')$$

$$r_> = \mathrm{Max}\,(r, r')$$

$$= \frac{r_<}{r_>}$$

Prob 4, Exam-2

(a)
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$
$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{q_a \, \delta^{(3)}(\vec{r}\,' - \vec{r}_a)}{|\vec{r} - \vec{r}\,'|}$$
$$= \frac{q_a}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_a|}$$

(b)
$$\vec{E} = -\vec{\nabla}\phi$$
$$= -\frac{q_a}{4\pi\epsilon_0} \vec{\nabla} \frac{1}{|\vec{r} - \vec{r}_a|}$$
$$= -\frac{q_a}{4\pi\epsilon_0} (-1) \frac{1}{|\vec{r} - \vec{r}_a|^3} (\vec{r} - \vec{r}_a)$$
$$= \frac{q_a}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_a}{|\vec{r} - \vec{r}_a|^3}$$

(c)
$$\vec{F} = q_b \, \vec{E}(\vec{r}_b)$$
$$= \frac{q_a q_b}{4\pi\epsilon_0} \frac{\vec{r}_b - \vec{r}_a}{|\vec{r}_b - \vec{r}_a|^3}$$